UNIT I: MATRICES

PART A:

1. Define characteristic equation, Eigen values and gives another name for Eigen values.

Answer:

Characteristic equation:

If 'A' is any square matrix of order 'n', then $|A - \lambda I| = 0$ is called the characteristic equation of 'A'.

Eigen values:

The roots of characteristic equation are called the Eigen values.

Another name for Eigen values:

- **Characteristic roots.**
- **Latent roots.**

2. State Cayley – Hamilton theorem and give two uses of Cayley – Hamilton theorem. [v ex]

[A/M 2008 R2008] [M/J 2010 R2008] [N/D 2014 R2013] [A/M 2015 R2008]

Answer:

Cayley – Hamilton theorem:

Every Square matrix satisfies its own characteristic Equation.

Uses:

Cayley – Hamilton theorem is useful to find the following

- **Inverse of matrix.**
- **Powers of matrix in terms of the lower powers of the matrix.**

3. Using Cayley Hamilton theorem, find the inverse of the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & -5 \end{bmatrix}$ [B]

Solution:

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & -5 \end{bmatrix}$

$S_1 = \text{Sum of diagonal elements}$

$= 2 - 5$

$\boxed{S_1 = -3}$

$S_2 = |A|$

$= \begin{vmatrix} 2 & 1 \\ 1 & -5 \end{vmatrix}$

$= -10 - 1$

$\boxed{S_2 = -11}$

The characteristic equation is $\lambda^2 - S_1\lambda + S_2 = 0$

$$\lambda^2 - (-3)\lambda + (-11) = 0$$

$$\lambda^2 + 3\lambda - 11 = 0$$

By Cayley – Hamilton theorem, $\boxed{A^2 + 3A - 11I = 0}$

Multiply by A^{-1}

$$A^{-1}A^2 + 3A^{-1}A - 11A^{-1}I = 0$$

$$A + 3I - 11A^{-1} = 0 \quad [\because A^{-1}A = I \text{ And } A^{-1}I = A^{-1}]$$

$$A + 3I = 11A^{-1}$$

$$11A^{-1} = A + 3I$$

$$11A^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & -5 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$11A^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & -5 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$11A^{-1} = \begin{bmatrix} 2+3 & 1+0 \\ 1+0 & -5+3 \end{bmatrix}$$

$$11A^{-1} = \begin{bmatrix} 5 & 1 \\ 1 & -2 \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{11}\begin{bmatrix} 5 & 1 \\ 1 & -2 \end{bmatrix}}$$

4. If $A = \begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix}$**, express** A^3 **in terms of A and I, using Cayley – Hamilton theorem. [B]**

Solution:

Let $A = \begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix}$

$S_1 = \text{Sum of diagonal elements}$

$= 1 + 5$

$\boxed{S_1 = 6}$

$S_2 = |A|$

$= \begin{vmatrix} 1 & 0 \\ 4 & 5 \end{vmatrix}$

$= 5 - 0$

$\boxed{S_2 = 5}$

The characteristic equation is $\lambda^2 - S_1\lambda + S_2 = 0$

$$\lambda^2 - 6\lambda + 5 = 0$$

By Cayley – Hamilton theorem, $\boxed{A^2 - 6A + 5I = 0}$

Multiply by A

$$A^3 - 6A^2 + 5A = 0$$

$$A^3 = 6A^2 - 5A$$

$$A^3 = 6[6A - 5I] - 5A \quad \{\because A^2 - 6A + 5I = 0 \Rightarrow A^2 = 6A - 5I\}$$

$$A^3 = 36A - 30I - 5A$$

$$\boxed{A^3 = 31A - 30I}$$

5. Define Eigen values and Eigenvector of a matrix. Also write properties of Eigen values.

Answer:

Eigen values and Eigenvector:

Let A be a square matrix. If there exists a scalar λ and a non zero column vector X such that AX = λX, then λ is called Eigen value of A and X is called Eigenvector of A.

Properties of Eigen values:

1. **Sum of the Eigen values = Sum of the main diagonal elements [Trace]**
2. **Product of the Eigen value = |A|, where A is the given matrix**
3. **Eigen values of A^T =Eigen values of A [Here A is a square matrix]**
4. **In a triangle matrix, the Eigen values are diagonal elements.**

6. Find the Eigen value of a matrix $\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}$ **corresponding to the Eigen vector** $\begin{bmatrix} -4 & -2 & 4 \end{bmatrix}^T$

[N/D 2015 R2008][A/M 2017 R2008]

Solution:

Given $A = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}$ $and\ X = \begin{bmatrix} -4 \\ -2 \\ 4 \end{bmatrix}$

$$AX = \lambda X$$

$$\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}\begin{bmatrix} -4 \\ -2 \\ 4 \end{bmatrix} = \lambda \begin{bmatrix} -4 \\ -2 \\ 4 \end{bmatrix}$$

$$-28 + 4 + 0 = -4\lambda$$

$$-24 = -4\lambda$$

$$\frac{-24}{-4} = \lambda$$

$$6 = \lambda$$

$$\boxed{\lambda = 6}$$

7. Obtain the Eigen values of A^3 **where** $A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$ **[B]**

Solution:

$S_1 = \text{Sum of diagonal elements}$

$= 3 + 2$

$\boxed{S_1 = 5}$

$S_2 = |A|$

$= \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix}$

$= 6 - 2$

$\boxed{S_2 = 4}$

The characteristic equation is $\lambda^2 - S_1\lambda + S_2 = 0$

$$\lambda^2 - 5\lambda + 4 = 0$$

$$(\lambda - 1)(\lambda - 4) = 0$$

$$\lambda = 1, \lambda = 4$$

The Eigen values of $A = 1, 4$

The Eigen values of $A^3 = 1^3, 4^3 = 1, 64$

8. If λ is an Eigen value of a square matrix A, prove that $k\lambda$ **is an Eigen value of** kA**. [M/J 2012 R2008]**

Answer:

Given λ is an Eigen value of a square matrix A

$\Rightarrow AX = \lambda X$ **and** $X \neq 0$

$\Rightarrow k(AX) = k(\lambda X)$

$\Rightarrow (kA)X = (k\lambda)X$

Hence $k\lambda$ **is an Eigen value of** kA**.**

9. If λ is an Eigen value of a square matrix A, prove that $\frac{1}{\lambda}$ **is an Eigen values of** A^{-1}**.**

(Or)

If λ is an Eigen value of a non-singular matrix A, show that λ^{-1} **is an Eigen values of** A^{-1}**.**

[B][G] [M/J 2012 R2008][N/D 2014 R2008] [N/D 2014 R2013]

Answer:

Given λ is an Eigen value of a square matrix A

$\Rightarrow AX = \lambda X$ and $X \neq 0$

$\Rightarrow A^{-1}(AX) = A^{-1}(\lambda X)$

$\Rightarrow \left(A^{-1}A\right)X = \lambda A^{-1}X$

$\Rightarrow I.X = \lambda A^{-1}X \quad [\because A^{-1}A = I]$

$\Rightarrow X = \lambda A^{-1}.X$

$\Rightarrow \frac{1}{\lambda}X = A^{-1}.X$

$\Rightarrow A^{-1}.X = \frac{1}{\lambda}X$

Hence $\frac{1}{\lambda}$ **is an Eigen value of** A^{-1} **.**

10. If λ is an Eigen value of a square matrix A, prove that λ^2 **is an Eigen values of** A^2 **.**

[M/J 2014 R2008] [Jan 2014 R2013][M/J 2016 R2013]

Answer:

Given λ is an Eigen value of a square matrix A

$\Rightarrow AX = \lambda X$ and $X \neq 0$

$\Rightarrow A(AX) = A(\lambda X)$

$\Rightarrow (AA)X = \lambda(AX)$

$\Rightarrow A^2X = \lambda(AX)$

$\Rightarrow A^2X = \lambda(\lambda X) \quad [\because AX = \lambda X]$

$\Rightarrow A^2X = \lambda^2X$

Hence λ^2 **is an Eigen value of** A^2 **.**

11. Prove that any square matrix A and its transpose A^T **have the same Eigen values. [G]**

(Or)

Prove that A and A^T **have the same Eigen values. [v ex]**

Answer:

$$\textit{Characteristic equation of } A = |A - \lambda I|$$

$$\textit{Characteristic equation of } A^T = \left[|A - \lambda I|\right]^T$$

$$= |A^T - \lambda I^T|$$

$$= |A - \lambda I| \qquad \left\{\because |A^T| = |A| \textit{ and } |I^T| = |I|\right\}$$

$$= \textit{The characteristic equation of } A$$

Therefore Characteristic equation of A = Characteristic equation of A^T

Hence Eigen values of A = Eigen values of A^T

12. Prove that the Eigen values of a triangular matrix are just the diagonal elements of the matrix. [G]

Answer:

Let $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$ **be any triangular matrix.**

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ 0 & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a_{nn} - \lambda \end{vmatrix} = 0$$

$$(a_{11} - \lambda)(a_{22} - \lambda)\dots(a_{nn} - \lambda) = 0$$

$$a_{11} - \lambda = 0, a_{22} - \lambda = 0, \dots a_{nn} - \lambda = 0$$

$$\boxed{a_{11} = \lambda}, \boxed{a_{22} = \lambda}, \dots \boxed{a_{nn} = \lambda}$$

Hence the Eigen values of a triangular matrix are just the diagonal elements of the matrix.

13. Prove that the characteristic roots of an idempotent matrix are either zero or unity. [B] [G]

Answer:

Given A is an idempotent matrix $\Rightarrow A^2 = A$

Let λ be an Eigen value of a square matrix A

$\Rightarrow AX = \lambda X$ **and** $X \neq 0$

$$\Rightarrow A(AX) = A(\lambda X)$$

$$\Rightarrow (AA)X = \lambda(AX)$$

$$\Rightarrow A^2 X = \lambda(\lambda X)$$

$$\Rightarrow AX = \lambda^2 X \quad \{\because A^2 = A\}$$

$$\Rightarrow \lambda X = \lambda^2 X \quad \{\because AX = \lambda X\}$$

$$\Rightarrow \lambda = \lambda^2$$

$$\Rightarrow \lambda - \lambda^2 = 0$$

$$\Rightarrow \lambda(1 - \lambda) = 0$$

$$\Rightarrow \lambda = 0, 1 - \lambda = 0$$

$\Rightarrow \lambda = 0, 1 = \lambda$

$\Rightarrow \boxed{\lambda = 0, \lambda = 1}$

14. Find the Sum and product of the Eigen values of the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ **[M/J 2014 R2008]**

<u>Solution:</u>

Let $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$.

Sum of the Eigen values = Sum of the main diagonal elements

$= -2 + 1 + 0$

$= -1$

Product of the Eigen value $= |A|$

$= \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$

$= -2\begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} - 2\begin{vmatrix} 2 & -6 \\ -1 & 0 \end{vmatrix} + (-3)\begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix}$

$= -2[0 - 12] - 2[0 - 6] - 3[-4 + 1]$

$= 45$

15. If the sum of two Eigen values and trace of 3×3 matrix A are equal, find the value of |A|. [v ex]

[A/M 2008 R2008][M/J 2009 R2008] [N/D 2016 R2013]

<u>Solution:</u>

Let $\lambda_1, \lambda_2, \lambda_3$ **be the three Eigen values of the matrix A.**

Given $\lambda_1 + \lambda_2 =$ **trace of a 3×3 matrix A**

$\lambda_1 + \lambda_2 =$ **Sum of the main diagonal elements**

$\lambda_1 + \lambda_2 =$ **Sum of the Eigen values**

$\lambda_1 + \lambda_2 = \lambda_1 + \lambda_2 + \lambda_3$

$\boxed{\lambda_3 = 0}$

WKT, Product of the Eigen value $= |A|$

$|A|$ **= Product of the Eigen value**

$$|A| = \lambda_1.\lambda_2.\lambda_3$$

$$|A| = \lambda_1.\lambda_2.0 \qquad \{\because \lambda_3 = 0\}$$

$$\boxed{|A| = 0}$$

16. The product of two Eigen values of the matrix A = $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ **is 16. Find the third Eigen value.**

[Jan 2012 R2008] [N/D 2013 R2007][Jan 2014 R2008] [M/J 2016 R2008, R2007] [N/D 2016 R2008]

<u>Solution:</u>

Given A = $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Let $\lambda_1, \lambda_2, \lambda_3$ **be the three Eigen values of the matrix A.**

Given $\lambda_1.\lambda_2 = 16$

WKT, Product of the Eigen value = $|A|$

$$\lambda_1.\lambda_2.\lambda_3 = |A|$$

$$\lambda_1.\lambda_2.\lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$16\lambda_3 = 6[9-1] - (-2)[-6+2] + 2[2-6]$$

$$16\lambda_3 = 32$$

$$\boxed{\lambda_3 = 2}.$$

17. Two Eigen values of the matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ **are 3 and 0. What is the third Eigen value and |A|?**

[A/M 2017 R2013][A/M 2018 R2017]

<u>Solution:</u>

Given $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Let $\lambda_1, \lambda_2, \lambda_3$ **are the Eigen values of A.**

Given $\lambda_1 = 3, \lambda_2 = 0$

Sum of the Eigen values = Sum of the main diagonal elements

$$\lambda_1 + \lambda_2 + \lambda_3 = 8 + 7 + 3$$

$$3 + 0 + \lambda_3 = 18$$

$$\lambda_3 = 18 - 3$$

$$\boxed{\lambda_3 = 15}$$

|A| = Product of the Eigen values = 3×0×15 =0

18. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$**, then find the Eigen value of** A^{-1} ***and*** A^{3} **[A/M 2015 R2008]**

Solution:

Given matrix is triangle matrix.

$\therefore$ **Eigen values of A** $= 1,4,6$

Eigen values of $A^{-1} = 1^{-1}, 4^{-1}, 6^{-1} = \frac{1}{1}, \frac{1}{4}, \frac{1}{6}$

Eigen values of $A^{3} = 1^{3}, 4^{3}, 6^{3} = 1, 64, 216$

19. Find the sum, product and sum of the squares of the Eigen values of A = $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ **[v ex]**

[M/J 2013 R2008] [A/M 2018 R2008]

Solution: **Given matrix is triangle matrix.**

$\therefore$**Eigen values of A = 3, 2, 5**

Sum of Eigen values of A = 3+2+5 = 10.

Product of Eigen values of A = 3×2×5 = 30.

Sum of the squares of the Eigen values of A = $3^2 + 2^2 + 5^2 = 9 + 4 + 25 = 38$

20. If 2, −1, −3 are Eigen values of the matrix A, and then find the Eigen values of $A^2 - 2I$ **.**

[M/J 2014 R2013][N/D 2015 R2008] [A/M 2017 R2008]

Solution:

Eigen values of $A = 2, -1, -3$

Eigen values of $A^2 - 2I = (2)^2 - 2,\ (-1)^2 - 2,\ (-3)^2 - 2$

$= 2, -1, 7$

21. If the Eigen values of the matrix A of order 3× 3 are 2, 3, and 1, then find the Eigen values of adjoint of A. **[Jan 2014 R2013] [M/J 2016 R2013]**

Solution:

Eigen values of $A = 2,3,1$

Eigen values of adj A = Eigen values of $|A| A^{-1}$

$= 6.2^{-1}, 6.3^{-1}, 6.1^{-1} \quad \{\because |A| = 2 \times 3 \times 1 = 6\}$

$= 6.\left(\frac{1}{2}\right), 6.\left(\frac{1}{3}\right), 6.\left(\frac{1}{1}\right)$

$$= 3,2,6$$

22. Show that the Eigen values of a null matrix are zero. [A/M 2018 R2017]

Answer:

Let a null matrix $A = \begin{vmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{vmatrix}$

$$\text{Characteristic equation } |A - \lambda I| = 0$$

$$\begin{vmatrix} 0-\lambda & 0 & \dots & 0 \\ 0 & 0-\lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0-\lambda \end{vmatrix} = 0$$

$$(-\lambda)^n = 0$$

$$-\lambda = 0$$

$$\boxed{\lambda = 0}$$

23. Can $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ be diagonalized. Why? [Jan 2010 R2008][Jan 2012 R2008]

Solution:

Yes, A can be diagonalized because Eigen values of A are 1, 1.

We can find distinct Eigen vector for 1, 1

24. State the condition for a quadratic form to be positive definite, positive semi definite, Negative definite, negative semi definite and indefinite [v ex]

Solution:

Condition	**Nature**
All the Eigen values are positive.	*Positive definite*
All the Eigen values are positive and at least one of the Eigen value is 0.	*Positive semi definite*
All the Eigen values are negative.	*Negative definite*
All the Eigen values are negative and at least one of the Eigen value is 0.	*Negative semi definite*
Positive as well as negative.	*indefinite*

25. Write down the matrix of the quadratic form $2x_1^2 + 5x_2^2 + 4x_1x_2 + 2x_3x_1$. [N/D 2016 R2013]

Solution:

Given Equation, $2x_1^2 + 5x_2^2 + 4x_1x_2 + 2x_3x_1$

$$Matrix\ A = \begin{bmatrix} coeff\ x_1^2 & \frac{1}{2}coeff\ x_1x_2 & \frac{1}{2}coeff\ x_1x_3 \\ \frac{1}{2}coeff\ x_1x_2 & coeff\ x_2^2 & \frac{1}{2}coeff\ x_2x_3 \\ \frac{1}{2}coeff\ x_1x_3 & \frac{1}{2}coeff\ x_2x_3 & coeff\ x_3^2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & \frac{1}{2}(4) & \frac{1}{2}(2) \\ \frac{1}{2}(4) & 5 & \frac{1}{2}(0) \\ \frac{1}{2}(2) & \frac{1}{2}(0) & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

26. Write the quadratic form for the matrix $\begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}$**. [M/J 2012 R2008][N/D 2013 R2007]**

Solution:

Given $\begin{matrix} & x_1 & x_2 & x_3 \\ x_1 & 0 & 5 & -1 \\ x_2 & 5 & 1 & 6 \\ x_3 & -1 & 6 & 2 \end{matrix}$

Q.F $= 0x_1^2 + 1x_2^2 + 2x_3^2 + 5x_1x_2 - 1x_1x_3 + 5x_1x_2 + 6x_2x_3 - 1x_1x_3 + 6x_2x_3$

$= x_2^2 + 2x_3^2 + 10x_1x_2 - 2x_1x_3 + 12x_2x_3$

27. Discuss the nature of the following Quadratic form $2x^2 + 3y^2 + 2z^2 + 2xy$

[Jan 2014 R2008] [N/D 2016 R2008]

Solution:

Given Equation, $2x^2 + 3y^2 + 2z^2 + 2xy$

$$The\ matrix\ of\ the\ Q.F = \begin{bmatrix} coeff\ x^2 & \frac{1}{2}coeff\ xy & \frac{1}{2}coeff\ xz \\ \frac{1}{2}coeff\ xy & coeff\ y^2 & \frac{1}{2}coeff\ yz \\ \frac{1}{2}coeff\ xz & \frac{1}{2}coeff\ zy & coeff\ z^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & \frac{1}{2}(2) & \frac{1}{2}(0) \\ \frac{1}{2}(2) & 3 & \frac{1}{2}(0) \\ \frac{1}{2}(0) & \frac{1}{2}(0) & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

To find the nature:

$$D_1 = |2| = 2 \ [+ve]$$

$$D_2 = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 6 - 1 = 5 \quad [+ve]$$

$$D_3 = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 2(6-0) - 1(2-0) + 0(0-0) = 10 \ [+ve]$$

The quadratic form is positive definite.

28. Determine the nature of the Quadratic forms $5x_1^2 + 5x_2^2 + 14x_3^2 + 2x_1x_2 - 16x_2x_3 - 8x_3x_1$ without reducing them to canonical form: [v]

Answer:

Given Equation $5x_1^2 + 5x_2^2 + 14x_3^2 + 2x_1x_2 - 16x_2x_3 - 8x_3x_1$

$$Matrix\ A = \begin{bmatrix} coeff\ x_1^2 & \frac{1}{2}coeff\ x_1x_2 & \frac{1}{2}coeff\ x_1x_3 \\ \frac{1}{2}coeff\ x_1x_2 & coeff\ x_2^2 & \frac{1}{2}coeff\ x_2x_3 \\ \frac{1}{2}coeff\ x_1x_3 & \frac{1}{2}coeff\ x_2x_3 & coeff\ x_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & \frac{1}{2}(2) & \frac{1}{2}(-8) \\ \frac{1}{2}(2) & 5 & \frac{1}{2}(-16) \\ \frac{1}{2}(-8) & \frac{1}{2}(-16) & 14 \end{bmatrix}$$

Matrix $A = \begin{bmatrix} 5 & 1 & -4 \\ 1 & 5 & -8 \\ -4 & -8 & 14 \end{bmatrix}$

To find nature:

$$D_1 = |5| = 5 \ [+ve]$$

$$D_2 = \begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} = 25 - 1 = 24 \ [+ve]$$

$$D_3 = \begin{vmatrix} 5 & 1 & -4 \\ 1 & 5 & -8 \\ -4 & -8 & 14 \end{vmatrix}$$

$$= 5\begin{vmatrix} 5 & -8 \\ -8 & 14 \end{vmatrix} - 1\begin{vmatrix} 1 & -8 \\ -4 & 14 \end{vmatrix} + (-4)\begin{vmatrix} 1 & 5 \\ -4 & -8 \end{vmatrix}$$

$$= 5[70 - 64] - [14 - 32] - 4[-8 + 20]$$

$$= 5[6] - [-18] - 4[12]$$

$$= 30 + 18 - 48$$

$$D_3 = 0$$

Nature = Positive semi definite.

29. Find the nature of the quadratic form $x^2 + 5y^2 + z^2 + 2xy + 2yz + 6zx$ **[G]**

Solution:

The matrix of the Q.F $= \begin{bmatrix} Coeff\ x^2 & \frac{1}{2}Coeff\ xy & \frac{1}{2}Coeff\ xz \\ \frac{1}{2}Coeff\ yx & Coeff\ y^2 & \frac{1}{2}Coeff\ yz \\ \frac{1}{2}Coeff\ zx & \frac{1}{2}Coeff\ zy & Coeff\ z^2 \end{bmatrix}$

$$= \begin{bmatrix} 1 & \frac{1}{2}(2) & \frac{1}{2}(6) \\ \frac{1}{2}(2) & 5 & \frac{1}{2}(2) \\ \frac{1}{2}(6) & \frac{1}{2}(2) & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

To find nature:

$D_1 = |1| = 1 \quad [+ve]$

$D_2 = \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} = 5 - 1 = 4 \quad [+ve]$

$$D_3 = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & 5 \\ 3 & 1 \end{vmatrix}$$

$$= [5 - 1] - [1 - 3] + 3[1 - 15]$$

$$= 4 + 2 - 42$$

$$= -36 \quad [-ve]$$

Nature = Indefinite.

30. Give the nature of the quadratic form whose matrix is $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ **[A/M 2015 R2013]**

<u>Solution:</u>

Given matrix is triangle matrix.

$\therefore$ **The Eigen values of A = -1, -1,-2**

Nature = Negative definite.

<u>PART B:</u>

1. Verify Cayley – Hamilton theorem and hence find A^{-1} *and* A^4 **if** $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$**. [B] [Jan 2014 R2013]**

<u>Answer:</u>

<u>To find Characteristic equation:</u>

$S_1 = Sum\ of\ diagonal\ elements$

$= 1 + 3 + 1$

$\boxed{S_1 = 5}$

$S_2 = Sum\ of\ minor\ of\ diagonal\ elements$

$$= \begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}$$

$$= [3 - 0] + [1 - 0] + [3 + 2]$$

$$= 3 + 1 + 5$$

$\boxed{S_2 = 9}$

$S_3 = |A|$

$$= \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} - 2\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} + (-2)\begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix}$$

$$= [3 - 0] - 2[-1 - 0] - 2[2 - 0]$$

$= 3 + 2 - 4$

$\boxed{S_3 = 1}$

The characteristic equation is $\boxed{\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0}$

$$\lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0$$

By Cayley – Hamilton theorem, $\boxed{A^3 - 5A^2 + 9A - I = 0}$

Verification:

$A^2 = A \times A$

$$A^2 = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 - 2 + 0 & 2 + 6 + 4 & -2 + 0 - 2 \\ -1 - 3 + 0 & -2 + 9 + 0 & 2 + 0 + 0 \\ 0 + 2 + 0 & 0 - 6 - 2 & 0 + 0 + 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}$$

$A^3 = A \times A^2$

$$A^3 = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \times \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -1 - 8 - 4 & 12 + 14 + 16 & -4 + 4 - 2 \\ 1 - 12 + 0 & -12 + 21 + 0 & 4 + 6 + 0 \\ 0 + 8 + 2 & 0 - 14 - 8 & 0 - 4 + 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix}$$

$$\text{Now } A^3 - 5A^2 + 9A - I = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - 5\begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - \begin{bmatrix} -5 & 60 & -20 \\ -20 & 35 & 10 \\ 10 & -40 & 5 \end{bmatrix} + \begin{bmatrix} 9 & 18 & -18 \\ -9 & 27 & 0 \\ 0 & -18 & 9 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -13 + 5 + 9 - 1 & 42 - 60 + 18 - 0 & -2 + 20 - 18 - 0 \\ -11 + 20 - 9 - 0 & 9 - 35 + 27 - 1 & 10 - 10 + 0 - 0 \\ 10 - 10 + 0 - 0 & -22 + 40 - 18 - 0 & -3 - 5 + 9 - 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= 0$$

<u>*To find A^{-1}:*</u>

$$A^3 - 5A^2 + 9A - I = 0$$

Multiply by A^{-1}

$$A^{-1}A^3 - 5A^{-1}A^2 + 9A^{-1}A - A^{-1}I = 0$$

$$A^2 - 5A + 9I - A^{-1} = 0$$

$$A^2 - 5A + 9I = A^{-1}$$

$$A^{-1} = A^2 - 5A + 9I$$

$$A^{-1} = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 10 & -10 \\ -5 & 15 & 0 \\ 0 & -10 & 5 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -1 - 5 + 9 & 12 - 10 + 0 & -4 + 10 + 0 \\ -4 + 5 + 0 & 7 - 15 + 9 & 2 - 0 + 0 \\ 2 - 0 + 0 & -8 + 10 + 0 & 1 - 5 + 9 \end{bmatrix}$$

$$\boxed{A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}}$$

<u>*To find A^4:*</u>

$$A^3 - 5A^2 + 9A - I = 0$$

Multiply by A

$$A^4 - 5A^3 + 9A^2 - A = 0$$

$$A^4 = 5A^3 - 9A^2 + A$$

$$A^4 = 5\begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - 9\begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} -65 & 210 & -10 \\ -55 & 45 & 50 \\ 50 & -110 & -15 \end{bmatrix} - \begin{bmatrix} -9 & 108 & -36 \\ -36 & 63 & 18 \\ 18 & -72 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} -65 + 9 + 1 & 210 - 108 + 2 & -10 + 36 - 2 \\ -55 + 36 - 1 & 45 - 63 + 3 & 50 - 18 + 0 \\ 50 - 18 + 0 & -110 + 72 - 2 & -15 - 9 + 1 \end{bmatrix}$$

$$\boxed{A^4 = \begin{bmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 32 & -40 & -23 \end{bmatrix}}$$

2. Verify Cayley –Hamilton theorem for the matrix $A=\begin{bmatrix}1&3&7\\4&2&3\\1&2&1\end{bmatrix}$ **and also use it to find** A^{-1} *and* A^4 **. [v]**

[N/D 2011 R2008] [A/M 2015 R2008] [N/D 2016 R2008]

<u>Answer:</u>

<u>To find Characteristic equation:</u>

$S_1 = Sum\ of\ diagonal\ elements$

$= 1+2+1$

$\boxed{S_1 = 4}$

$S_2 = Sum\ of\ minor\ of\ diagonal\ elements$

$= \begin{vmatrix}2&3\\2&1\end{vmatrix} + \begin{vmatrix}1&7\\1&1\end{vmatrix} + \begin{vmatrix}1&3\\4&2\end{vmatrix}$

$= [2-6]+[1-7]+[2-12]$

$= -4-6-10$

$\boxed{S_2 = -20}$

$S_3 = |A|$

$= \begin{vmatrix}1&3&7\\4&2&3\\1&2&1\end{vmatrix}$

$= 1\begin{vmatrix}2&3\\2&1\end{vmatrix} - 3\begin{vmatrix}4&3\\1&1\end{vmatrix} + 7\begin{vmatrix}4&2\\1&2\end{vmatrix}$

$= [2-6]-3[4-3]+7[8-2]$

$= -4-3+42$

$\boxed{S_3 = 35}$

The characteristic equation is $\boxed{\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0}$

$$\lambda^3 - 4\lambda^2 + (-20)\lambda - 35 = 0$$

$$\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$$

By Cayley – Hamilton theorem, $A^3 - 4A^2 - 20A - 35I = 0$

<u>Verification:</u>

$$A^2 = A \times A$$

$$A^2 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1+12+7 & 3+6+14 & 7+9+7 \\ 4+8+3 & 12+4+6 & 28+6+3 \\ 1+8+1 & 3+4+2 & 7+6+1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$A^3 = A \times A^2$$

$$A^3 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 20+45+70 & 23+66+63 & 23+111+98 \\ 80+30+30 & 92+44+27 & 92+74+42 \\ 20+30+10 & 23+44+9 & 23+74+14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}$$

Now $A^3 - 4A^2 - 20A - 35I = \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - 4\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 20\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 35\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - \begin{bmatrix} 80 & 92 & 92 \\ 60 & 88 & 148 \\ 40 & 36 & 56 \end{bmatrix} - \begin{bmatrix} 20 & 60 & 140 \\ 80 & 40 & 60 \\ 20 & 40 & 20 \end{bmatrix} - \begin{bmatrix} 35 & 0 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 35 \end{bmatrix}$$

$$= \begin{bmatrix} 135-80-20-35 & 152-92-60-0 & 232-92-140-0 \\ 140-60-80-0 & 163-88-40-35 & 208-148-60-0 \\ 60-40-20-0 & 76-36-40-0 & 111-56-20-35 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= 0$$

To find A^{-1}:

$$A^3 - 4A^2 - 20A - 35I = 0$$

Multiply by A^{-1}

$$A^{-1}A^3 - 4A^{-1}A^2 - 20A^{-1}A - 35A^{-1}I = 0$$

$$A^2 - 4A - 20I - 35A^{-1} = 0$$

$$A^2 - 4A - 20I = 35A^{-1}$$

$$35A^{-1} = A^2 - 4A - 20I$$

$$35A^{-1} = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 4\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 20\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$35A^{-1} = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - \begin{bmatrix} 4 & 12 & 28 \\ 16 & 8 & 12 \\ 4 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix}$$

$$35A^{-1} = \begin{bmatrix} 20-4-20 & 23-12-0 & 23-28-0 \\ 15-16-0 & 22-8-20 & 37-12-0 \\ 10-4-0 & 9-8-0 & 14-4-20 \end{bmatrix}$$

$$35A^{-1} = \begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}}$$

To find A^4:

$$A^3 - 4A^2 - 20A - 35I = 0$$

Multiply by A

$$A^4 - 4A^3 - 20A^2 - 35A = 0$$

$$A^4 = 4A^3 + 20A^2 + 35A$$

$$A^4 = 4\begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} + 20\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} + 35\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 540 & 608 & 928 \\ 560 & 652 & 832 \\ 240 & 304 & 444 \end{bmatrix} + \begin{bmatrix} 400 & 460 & 460 \\ 300 & 440 & 740 \\ 200 & 180 & 280 \end{bmatrix} + \begin{bmatrix} 35 & 105 & 245 \\ 140 & 70 & 105 \\ 35 & 70 & 35 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 540+400+35 & 608+460+105 & 928+460+245 \\ 560+300+140 & 652+440+70 & 832+740+105 \\ 240+200+35 & 304+180+70 & 444+280+35 \end{bmatrix}$$

$$\boxed{A^4 = \begin{bmatrix} 975 & 1173 & 1633 \\ 1000 & 1162 & 1677 \\ 475 & 554 & 759 \end{bmatrix}}$$

3. **Verify that the matrix** A= $\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ **satisfies its characteristic equation. Also find** $A^{-1} and\ A^4$ **. [v]**

[M/J 2010 R2008][M/J 2013 R2008][N/D 2013 R2007] [N/D 2014 R2013][A/M 2017 R2013]

Answer:

To find Characteristic equation:

$S_1 = Sum\ of\ diagonal\ elements$

$= 2 + 2 + 2$

$\boxed{S_1 = 6}$

$S_2 = Sum\ of\ minor\ of\ diagonal\ elements$

$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$

$= [4 - 1] + [4 - 2] + [4 - 1]$

$= 3 + 2 + 3$

$\boxed{S_2 = 8}$

$S_3 = |A|$

$= \begin{vmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$

$= 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix}$

$= 2[4 - 1] + [-2 + 1] + 2[1 - 2]$

$= 6 - 1 - 2$

$\boxed{S_3 = 3}$

The characteristic equation is $\boxed{\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0}$

$$\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$$

By Cayley – Hamilton theorem, $A^3 - 6A^2 + 8A - 3I = 0$

Verification:

$A^2 = A \times A$

$A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$A^2 = \begin{bmatrix} 4+1+2 & -2-2-2 & 4+1+4 \\ -2-2-1 & 1+4+1 & -2-2-2 \\ 2+1+2 & -1-2-2 & 2+1+4 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = A \times A^2$$

$$A^3 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \times \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 14+5+10 & -12-6-10 & 18+6+14 \\ -7-10-5 & 6+12+5 & -9-12-7 \\ 7+5+10 & -6-6-10 & 9+6+14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

$$\text{Now } A^3 - 6A^2 + 8A - 3I = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - 6\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} + 8\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - \begin{bmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{bmatrix} + \begin{bmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 29-42+16-3 & -28+36-8-0 & 38-54+16-0 \\ -22+30-8-0 & 23-36+16-3 & -28+36-8-0 \\ 22-30+8-0 & -22+30-8-0 & 29-42+16-3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= 0$$

To find A^{-1}:

$$A^3 - 6A^2 + 8A - 3I = 0$$

Multiply by A^{-1}

$$A^{-1}A^3 - 6A^{-1}A^2 + 8A^{-1}A - 3A^{-1}I = 0$$

$$A^2 - 6A + 8I - 3A^{-1} = 0$$

$$A^2 - 6A + 8I = 3A^{-1}$$

$$3A^{-1} = A^2 - 6A + 8I$$

$$3A^{-1} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$3A^{-1} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 12 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$3A^{-1} = \begin{bmatrix} 7-12+8 & -6+6+0 & 9-12+0 \\ -5+6+0 & 6-12+8 & -6+6+0 \\ 5-6+0 & -5+6+0 & 7-12+8 \end{bmatrix}$$

$$3A^{-1} = \begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}}$$

<u>*To find* A^4 :</u>

$$A^3 - 6A^2 + 8A - 3I = 0$$

Multiply by A

$$A^4 - 6A^3 + 8A^2 - 3A = 0$$

$$A^4 = 6A^3 - 8A^2 + 3A$$

$$A^4 = 6\begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - 8\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} + 3\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 174 & -168 & 228 \\ -132 & 138 & -168 \\ 132 & -132 & 174 \end{bmatrix} - \begin{bmatrix} 56 & -48 & 72 \\ -40 & 48 & -48 \\ 40 & -40 & 56 \end{bmatrix} + \begin{bmatrix} 6 & -3 & 6 \\ -3 & 6 & -3 \\ 3 & -3 & 6 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 174-56+6 & -168+48-3 & 228-72+6 \\ -132+40-3 & 138-48+6 & -168+48-3 \\ 132-40+3 & -132+40-3 & 174-56+6 \end{bmatrix}$$

$$\boxed{A^4 = \begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}}$$

4. Using Cayley –Hamilton theorem, find A^{-1} *and* A^4 **for the matrix** $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$.

[M/J 2014 R2013] [Jan 2014 R2008] [N/D 2015 R2008][A/M 2017 R2008]

<u>**Answer:**</u>

<u>**To find Characteristic equation:**</u>

$S_1 = Sum\ of\ diagonal\ elements$

$= 1 + 1 + 1$

$\boxed{S_1 = 3}$

$S_2 = Sum\ of\ minor\ of\ diagonal\ elements$

$$= \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}$$

$= [1 - 1] + [1 - 3] + [1 - 0]$

$= 0 - 2 + 1$

$\boxed{S_2 = -1}$

$S_3 = |A|$

$$= \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} - 0\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$$

$= [1 - 1] - 0 + 3[-2 - 1]$

$= 3[-3]$

$\boxed{S_3 = -9}$

The characteristic equation is $\boxed{\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0}$

$$\lambda^3 - 3\lambda^2 + (-1)\lambda - (-9) = 0$$

$$\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$$

By Cayley – Hamilton theorem, $\boxed{A^3 - 3A^2 - A + 9I = 0}$

<u>$To\ find\ A^{-1}$</u>:

$A^3 - 3A^2 - A + 9I = 0$

Multiply by A^{-1}

$A^{-1}A^3 - 3A^{-1}A^2 - A^{-1}A + 9A^{-1}I = 0$

$A^2 - 3A - I + 9A^{-1} = 0$

$9A^{-1} = -A^2 + 3A + I$

$$9A^{-1} = -\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$9A^{-1} = -\begin{bmatrix} 1+0+3 & 0+0-3 & 3+0+3 \\ 2+2-1 & 0+1+1 & 6-1-1 \\ 1-2+1 & 0-1-1 & 3+1+1 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 9 \\ 6 & 3 & -3 \\ 3 & -3 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$9A^{-1} = -\begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 9 \\ 6 & 3 & -3 \\ 3 & -3 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$9A^{-1} = \begin{bmatrix} -4 & 3 & -6 \\ -3 & -2 & -4 \\ 0 & 2 & -5 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 9 \\ 6 & 3 & -3 \\ 3 & -3 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$9A^{-1} = \begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}}$$

To find A^{-1}:

$$A^3 - 3A^2 - A + 9I = 0$$

Multiply by A

$$A^4 - 3A^3 - A^2 + 9A = 0$$

$$A^4 = 3A^3 + A^2 - 9A$$

$$A^4 = 3\left[3A^2 + A - 9I\right] + A^2 - 9A \qquad \left\{\begin{array}{l} \because A^3 - 3A^2 - A + 9I = 0 \\ \Rightarrow A^3 = 3A^2 + A - 9I \end{array}\right\}$$

$$A^4 = 9A^2 + 3A - 27I + A^2 - 9A$$

$$A^4 = 10A^2 - 6A - 27I$$

$$A^4 = 10\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} - 27\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^4 = 10\begin{bmatrix} 1+0+3 & 0+0-3 & 3+0+3 \\ 2+2-1 & 0+1+1 & 6-1-1 \\ 1-2+1 & 0-1-1 & 3+1+1 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} - 27\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^4 = 10\begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} - 27\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 40 & -30 & 60 \\ 30 & 20 & 40 \\ 0 & -20 & 50 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 18 \\ 12 & 6 & -6 \\ 6 & -6 & 6 \end{bmatrix} - \begin{bmatrix} 27 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 40-6-27 & -30-0-0 & 60-18-0 \\ 30-12-0 & 20-6-27 & 40+6-0 \\ 0-6-0 & -20+6-0 & 50-6-27 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 7 & -30 & 42 \\ 18 & -13 & 46 \\ -6 & -14 & 17 \end{bmatrix}$$

5. Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$**. Also find its inverse and** A^4 **. [G]**

Answer:

To find Characteristic equation:

$S_1 = Sum\ of\ diagonal\ elements$

$= 1 + 3 - 4$

$\boxed{S_1 = 0}$

$S_2 = Sum\ of\ minor\ of\ diagonal\ elements$

$= \begin{vmatrix} 3 & -3 \\ -4 & -4 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}$

$= [-12 - 12] + [-4 + 6] + [3 - 1]$

$= -24 + 2 + 2$

$\boxed{S_2 = -20}$

$S_3 = |A|$

$= \begin{vmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{vmatrix}$

$= 1\begin{vmatrix} 3 & -3 \\ -4 & -4 \end{vmatrix} - 1\begin{vmatrix} 1 & -3 \\ -2 & -4 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix}$

$= [-12 - 12] - [-4 - 6] + 3[-4 + 6]$

$= -24 + 10 + 6$

$\boxed{S_3 = -8}$

The characteristic equation is $\boxed{\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0}$

$$\lambda^3 - 0\lambda^2 + (-20)\lambda - (-8) = 0$$

$$\lambda^3 - 20\lambda + 8 = 0$$

By Cayley – Hamilton theorem, $\boxed{A^3 - 20A + 8I = 0}$

To find A^{-1}:

$A^3 - 20A + 8I = 0$

Multiply by A^{-1}

$$A^{-1}A^{3} - 20A^{-1}A + 8A^{-1}I = 0$$

$$A^{2} - 20I + 8A^{-1} = 0$$

$$8A^{-1} = -A^{2} + 20I$$

$$8A^{-1} = -\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} + 20\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$8A^{-1} = -\begin{bmatrix} 1+1-6 & 1+3-12 & 3-3-12 \\ 1+3+6 & 1+9+12 & 3-9+12 \\ -2-4+8 & -2-12+16 & -6+12+16 \end{bmatrix} + \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix}$$

$$8A^{-1} = -\begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix} + \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix}$$

$$8A^{-1} = \begin{bmatrix} 4 & 8 & 12 \\ -10 & -22 & -6 \\ -2 & -2 & -22 \end{bmatrix} + \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix}$$

$$8A^{-1} = \begin{bmatrix} 4+20 & 8+0 & 12+0 \\ -10+0 & -22+20 & -6+0 \\ -2+0 & -2+0 & -22+20 \end{bmatrix}$$

$$8A^{-1} = \begin{bmatrix} 24 & 8 & 12 \\ -10 & -2 & -6 \\ -2 & -2 & -2 \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{8}\begin{bmatrix} 24 & 8 & 12 \\ -10 & -2 & -6 \\ -2 & -2 & -2 \end{bmatrix}}$$

<u>To find A^{4} :</u>

$$A^{3} - 20A + 8I = 0$$

Multiply by A

$$A^{4} - 20A^{2} + 8A = 0$$

$$A^{4} = 20A^{2} - 8A$$

$$A^{4} = 20\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} - 8\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

$$A^{4} = 20\begin{bmatrix} 1+1-6 & 1+3-12 & 3-3-12 \\ 1+3+6 & 1+9+12 & 3-9+12 \\ -2-4+8 & -2-12+16 & -6+12+16 \end{bmatrix} - 8\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

$$A^4 = 20\begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix} - 8\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} -80 & -160 & -240 \\ 200 & 440 & 120 \\ 40 & 40 & 440 \end{bmatrix} - \begin{bmatrix} 8 & 8 & 24 \\ 8 & 24 & -24 \\ -16 & -32 & -32 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} -80-8 & -160-8 & -240-24 \\ 200-8 & 440-24 & 120+24 \\ 40+16 & 40+32 & 440+32 \end{bmatrix}$$

$$\boxed{A^4 = \begin{bmatrix} -88 & -168 & -264 \\ 192 & 416 & 144 \\ 56 & 72 & 472 \end{bmatrix}}$$

6. Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$**. Also find the matrix represented by** $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$ **. [v] [G] [B]**

[M/J 2009 R2008][Jan 2010 Tirunelveli] [N/D 2015 R2013]

<u>Answer:</u>

<u>To find Characteristic equation:</u>

$S_1 = Sum\ of\ diagonal\ elements$

$= 2 + 1 + 2$

$\boxed{S_1 = 5}$

$S_2 = Sum\ of\ minor\ of\ diagonal\ elements$

$= \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$

$= [2 - 0] + [4 - 1] + [2 - 0]$

$= 2 + 3 + 2$

$\boxed{S_2 = 7}$

$S_3 = |A|$

$$= \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= 2[2-0] - 1[0-0] + 1[0-1]$$

$$= 4 - 0 - 1$$

$\boxed{S_3 = 3}$

The characteristic equation is $\boxed{\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0}$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley – Hamilton theorem, $\boxed{A^3 - 5A^2 + 7A - 3I = 0} \rightarrow (*)$

To find polynomial:

$$\begin{array}{r|l} & A^5 + A \\ \hline A^3 - 5A^2 + 7A - 3I & A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I \\ & A^8 - 5A^7 + 7A^6 - 3A^5 \\ & - \quad + \quad - \quad + \\ \hline & 0 + A^4 - 5A^3 + 8A^2 - 2A \\ & A^4 - 5A^3 + 7A^2 - 3A \\ & - \quad + \quad - \quad + \\ \hline & A^2 + A + I \end{array}$$

Now $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$$= \left[A^3 - 5A^2 + 7A - 3I\right]\left[A^5 + A\right] + A^2 + A + I$$

$$= [0]\left[A^5 + A\right] + A^2 + A + I \quad [\because by(*)]$$

$$= A^2 + A + I$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+1 & 2+1+1 & 2+0+2 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 2+0+2 & 1+1+2 & 1+0+4 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

7. Verify Cayley Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$. Also find its inverse, A^3 and express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A. [G]

Solution:

Given $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

S_1 = Sum of diagonal elements

$= 1 + 3$

$\boxed{S_1 = 4}$

$S_2 = |A|$

$= \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix}$

$= 3 - 8$

$\boxed{S_2 = -5}$

The characteristic equation is $\boxed{\lambda^2 - S_1\lambda + S_2 = 0}$

$$\lambda^2 - 4\lambda + (-5) = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

By Cayley – Hamilton theorem, $\boxed{A^2 - 4A - 5I = 0} \rightarrow (*)$

Verification:

$$A^2 = A \times A$$

$$A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1+8 & 4+12 \\ 2+6 & 8+9 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 16-16-0 \\ 8-8-0 & 17-12-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= 0$$

To find A^{-1}:

$$A^2 - 4A - 5I = 0$$

Multiply by A^{-1}

$$A^{-1}A^2 - 4A^{-1}A - 5A^{-1}I = 0$$

$$A - 4I - 5A^{-1} = 0$$

$$A - 4I = 5A^{-1}$$

$$5A^{-1} = A - 4I$$

$$5A^{-1} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$5A^{-1} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$5A^{-1} = \begin{bmatrix} 1-4 & 4-0 \\ 2-0 & 3-4 \end{bmatrix}$$

$$5A^{-1} = \begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}}$$

To find A^3:

$$A^2 - 4A - 5I = 0$$

Multiply by A

$$A^3 - 4A^2 - 5A = 0$$

$$A^3 = 4A^2 + 5A$$

$$A^3 = 4\begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} + 5\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 36 & 64 \\ 32 & 68 \end{bmatrix} + \begin{bmatrix} 5 & 20 \\ 10 & 15 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 36+5 & 64+20 \\ 32+10 & 68+15 \end{bmatrix}$$

$$\boxed{A^3 = \begin{bmatrix} 41 & 84 \\ 42 & 83 \end{bmatrix}}$$

__To find polynomial__:

$$\begin{array}{r|l} & A^3 - 2A + 3 \\ \hline A^2 - 4A - 5I & A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I \\ & A^5 - 4A^4 - 5A^3 \\ & (-)\quad (+)\quad (+) \\ \hline & \qquad -2A^3 + 11A^2 - A \\ & \qquad -2A^3 + 8A^2 + 10A \\ & \qquad (+)\quad (-)\quad (-) \\ \hline & \qquad\qquad 3A^2 - 11A - 10I \\ & \qquad\qquad 3A^2 - 12A - 15I \\ & \qquad\qquad (-)\quad (+)\quad (+) \\ \hline & \qquad\qquad\qquad A + 5I \end{array}$$

Now $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = \left[A^3 - 2A + 3\right]\left[A^2 - 4A - 5I\right] + A + 5I$

$$= \left[A^3 - 2A + 3\right][0] + A + 5I \qquad [\because by(*)]$$

$$= A + 5I$$

8. Find the Eigen values of A and using Cayley – Hamilton theorem, find A^n **, given that A=** $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$**.**

[B][v][Jan 2012 R2008][M/J 2014 R2008]

__Solution:__

Given $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

S_1 = *Sum of diagonal elements*

$$= 1 + 3$$

$$\boxed{S_1 = 4}$$

$S_2 = |A|$

$= \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix}$

$= 3 - 8$

$\boxed{S_2 = -5}$

The characteristic equation is $\lambda^2 - S_1\lambda + S_2 = 0$

$\lambda^2 - 4\lambda + (-5) = 0$

$\lambda^2 - 4\lambda - 5 = 0$

By Cayley – Hamilton theorem, $\boxed{A^2 - 4A - 5I = 0} \rightarrow (*)$

<u>To find Eigen values:</u>

$\lambda^2 - 4\lambda - 5 = 0$

$(\lambda + 1)(\lambda - 5) = 0 \quad \begin{bmatrix} \because sum = -4 \text{ and } product = -5 \\ (1, -5) \end{bmatrix}$

$\lambda = -1, \lambda = 5$

Eigen values are -1, 5.

Let $\boxed{\lambda^n = \left[\lambda^2 - 4\lambda - 5\right]Q(\lambda) + [a\lambda + b]} \rightarrow (1)$

Put $\lambda = -1$ **in** (1)

$(-1)^n = [0]Q(-1) + [a(-1) + b]$

$(-1)^n = -a + b$

$-a + b = (-1)^n \rightarrow (2)$

Put $\lambda = 5$ **in** (1)

$(5)^n = [0]Q(5) + [a(5) + b]$

$5^n = 5a + b$

$5a + b = 5^n \rightarrow (3)$

<u>*Solving* (2) *and* (3):</u>

$(2) \times 5 \Rightarrow -5a + 5b = 5(-1)^n$

$(3) \quad \Rightarrow 5a + b = 5^n$

$6b = 5^n + 5(-1)^n$

$$\boxed{b = \frac{5^n + 5(-1)^n}{6}}$$

Put 'b' value in (2)

$$-a + \frac{5^n + 5(-1)^n}{6} = (-1)^n$$

$$\frac{5^n + 5(-1)^n}{6} - (-1)^n = a$$

$$\frac{5^n + 5(-1)^n - 6(-1)^n}{6} = a$$

$$\frac{5^n - (-1)^n}{6} = a$$

$$\boxed{a = \frac{5^n - (-1)^n}{6}}$$

Put $\lambda = A$ **in** (1)

$$A^n = \left[A^2 - 4A - 5\right]Q(A) + \left[aA + bI\right]$$

$$A^n = [0]Q(A) + [aA + bI] \qquad \{\because by(*)\}$$

$$A^n = aA + bI$$

$$\boxed{A^n = \left[\frac{5^n - (-1)^n}{6}\right]\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \left[\frac{5^n + 5(-1)^n}{6}\right]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

9. Find the Eigen values and Eigen vectors of $\begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}$ **[v]**

[N/D 2011 R2008] [N/D 2016 R2013] [A/M 2018 R2017]

Answer:

Matrix $A = \begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}$

To find Characteristic equation:

$S_1 = Sum\ of\ diagonal\ elements$

$= 11 - 2 - 6$

$\boxed{S_1 = 3}$

$S_2 = Sum\ of\ minor\ of\ diagonal\ elements$

$$= \begin{vmatrix} -2 & -5 \\ -4 & -6 \end{vmatrix} + \begin{vmatrix} 11 & -7 \\ 10 & -6 \end{vmatrix} + \begin{vmatrix} 11 & -4 \\ 7 & -2 \end{vmatrix}$$

$$= [12 - 20] + [-66 + 70] + [-22 + 28]$$

$$= -8 + 4 + 6$$

$$\boxed{S_2 = 2}$$

$$S_3 = |A|$$

$$= \begin{vmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{vmatrix}$$

$$= 11\begin{vmatrix} -2 & -5 \\ -4 & -6 \end{vmatrix} - (-4)\begin{vmatrix} 7 & -5 \\ 10 & -6 \end{vmatrix} + (-7)\begin{vmatrix} 7 & -2 \\ 10 & -4 \end{vmatrix}$$

$$= 11[12 - 20] + 4[-42 + 50] - 7[-28 + 20]$$

$$= 11[-8] + 4[8] - 7[-8]$$

$$= -88 + 32 + 56$$

$$\boxed{S_3 = 0}$$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$$\lambda^3 - 3\lambda^2 + 2\lambda - 0 = 0$$

$$\lambda^3 - 3\lambda^2 + 2\lambda = 0$$

<u>To find Eigen values:</u>

$$\lambda^3 - 3\lambda^2 + 2\lambda = 0$$

$$\lambda\left(\lambda^2 - 3\lambda + 2\right) = 0$$

$$\lambda = 0 \;\; or \; \lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 1)(\lambda - 2) = 0 \quad \begin{bmatrix} \because sum = -3 \; and \; product = 2 \\ (-1, -2) \end{bmatrix}$$

$$\lambda = 1, \lambda = 2$$

Eigen values are 0, 1, 2

<u>To find Eigen vector:</u>

<u>Case (i) λ = 0:</u>

$$[A - \lambda I]X = 0$$

$$[A - 0.I]X = 0 \quad \{\because \lambda = 0\}$$

$$AX = 0$$

$$\begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Select last two rows

x_1		x_2		x_3
-2	-5	7	-2	
-4	-6	10	-4	

$$\frac{x_1}{12 - 20} = \frac{x_2}{-50 + 42} = \frac{x_3}{-28 + 20}$$

$$\frac{x_1}{-8} = \frac{x_2}{-8} = \frac{x_3}{-8}$$

Divided by (-8)

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Eigen vector $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

<u>Case (ii) λ = 1:</u>

$$\left[A - \lambda I\right]X = 0$$

$$\left[A - 1.I\right]X = 0 \quad \{\because \lambda = 1\}$$

$$\begin{bmatrix} 11 - 1 & -4 & -7 \\ 7 & -2 - 1 & -5 \\ 10 & -4 & -6 - 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 10 & -4 & -7 \\ 7 & -3 & -5 \\ 10 & -4 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Select last two rows

x_1		x_2		x_3
-3	-5	7	-3	
-4	-7	10	-4	

$$\frac{x_1}{21 - 20} = \frac{x_2}{-50 + 49} = \frac{x_3}{-28 + 30}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{2}$$

Eigen vector $X_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

<u>Case (iii) λ = 2:</u>

$$\left[A - \lambda I\right]X = 0$$

$$\left[A - 2.I\right]X = 0 \quad \{\because \lambda = 2\}$$

$$\begin{bmatrix} 11-2 & -4 & -7 \\ 7 & -2-2 & -5 \\ 10 & -4 & -6-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 9 & -4 & -7 \\ 7 & -4 & -5 \\ 10 & -4 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Select last two rows

$$\begin{matrix} & x_1 & & x_2 & & x_3 & \\ -4 & & -5 & & 7 & & -4 \\ -4 & & -8 & & 10 & & -4 \end{matrix}$$

$$\frac{x_1}{32-20} = \frac{x_2}{-50+56} = \frac{x_3}{-28+40}$$

$$\frac{x_1}{12} = \frac{x_2}{6} = \frac{x_3}{12}$$

Divided by 6

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{2}$$

Eigen vector $X_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

10. Find the Eigen values and Eigen vectors of $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ **[B] [Jan 2014 R2008] [A/M 2017 R2008]**

<u>Answer:</u>

Matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

<u>To find Characteristic equation:</u>

$S_1 = \text{Sum of diagonal elements}$

$= 8 + 7 + 3$

$\boxed{S_1 = 18}$

$S_2 = \text{Sum of minor of diagonal elements}$

$$= \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$= [21 - 16] + [24 - 4] + [56 - 36]$

$= 5 + 20 + 20$

$\boxed{S_2 = 45}$

$S_3 = |A|$

$$= \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$$= 8\begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} - (-6)\begin{vmatrix} -6 & -4 \\ 2 & 3 \end{vmatrix} + 2\begin{vmatrix} -6 & 7 \\ 2 & -4 \end{vmatrix}$$

$$= 8[21 - 16] + 6[-18 + 8] + 2[24 - 14]$$

$$= 8[5] + 6[-10] + 2[10]$$

$$= 40 - 60 + 20$$

$$\boxed{S_3 = 0}$$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$$\lambda^3 - 18\lambda^2 + 45\lambda - 0 = 0$$

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

<u>To find Eigen values:</u>

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\lambda\left(\lambda^2 - 18\lambda + 45\right) = 0$$

$$\lambda = 0 \;\; or \;\; \lambda^2 - 18\lambda + 45 = 0$$

$$(\lambda - 3)(\lambda - 15) = 0 \quad \begin{bmatrix} \because sum = -18 \; and \; product = 45 \\ (-3, -15) \end{bmatrix}$$

$$\lambda = 3, \lambda = 15$$

Eigen values are 0, 3, 15

<u>To find Eigen vector:</u>

<u>Case (i) λ = 0:</u>

$$[A - \lambda I]X = 0$$

$$[A - 0.I]X = 0 \quad \{\because \lambda = 0\}$$

$$AX = 0$$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Select last two rows

$$\begin{matrix} & x_1 & & x_2 & & x_3 & \\ 7 & & -4 & & -6 & & 7 \\ -4 & & 3 & & 2 & & -4 \end{matrix}$$

$$\frac{x_1}{21 - 16} = \frac{x_2}{-8 + 18} = \frac{x_3}{24 - 14}$$

$$\frac{x_1}{5}=\frac{x_2}{10}=\frac{x_3}{10}$$

Divided by 5

$$\frac{x_1}{1}=\frac{x_2}{2}=\frac{x_3}{2}$$

Eigen vector $X_1=\begin{bmatrix}1\\2\\2\end{bmatrix}$

<u>Case (ii) λ = 3:</u>

$$[A-\lambda I]X=0$$

$$[A-3I]X=0 \quad \{\because \lambda=3\}$$

$$\begin{bmatrix}8-3 & -6 & 2\\ -6 & 7-3 & -4\\ 2 & -4 & 3-3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$$

$$\begin{bmatrix}5 & -6 & 2\\ -6 & 4 & -4\\ 2 & -4 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$$

Select last two rows

$$\begin{matrix} & x_1 & & x_2 & & x_3 & \\ 4 & & -4 & & -6 & & 4\\ -4 & & 0 & & 2 & & -4\end{matrix}$$

$$\frac{x_1}{0-16}=\frac{x_2}{-8-0}=\frac{x_3}{24-8}$$

$$\frac{x_1}{-16}=\frac{x_2}{-8}=\frac{x_3}{16}$$

Divided by (−8)

$$\frac{x_1}{2}=\frac{x_2}{1}=\frac{x_3}{-2}$$

Eigen vector $X_2=\begin{bmatrix}2\\1\\-2\end{bmatrix}$

<u>Case (iii) λ = 15:</u>

$$[A-\lambda I]X=0$$

$$[A-15I]X=0 \quad \{\because \lambda=15\}$$

$$\begin{bmatrix}8-15 & -6 & 2\\ -6 & 7-15 & -4\\ 2 & -4 & 3-15\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$$

$$\begin{bmatrix}-7 & -6 & 2\\ -6 & -8 & -4\\ 2 & -4 & -12\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$$

Select last two rows

$$\begin{matrix} & x_1 & x_2 & x_3 \\ -8 & -4 & -6 & -8 \\ -4 & -12 & 2 & -4 \end{matrix}$$

$$\frac{x_1}{96-16} = \frac{x_2}{-8-72} = \frac{x_3}{24+16}$$

$$\frac{x_1}{80} = \frac{x_2}{-80} = \frac{x_3}{40}$$

Divided by 40

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

Eigen vector $X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

11. Find the Eigen values and Eigen vectors of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ **[v] [G]**

[N/D 2010 R2004] [N/D 2014 R2008] [M/J 2016 R2013]

Answer:

Matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

To find Characteristic equation:

$S_1 = \text{Sum of diagonal elements}$

$= 1 + 5 + 1$

$\boxed{S_1 = 7}$

$S_2 = \text{Sum of minor of diagonal elements}$

$= \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix}$

$= [5 - 1] + [1 - 9] + [5 - 1]$

$= 4 - 8 + 4$

$\boxed{S_2 = 0}$

$S_3 = |A|$

$= \begin{vmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{vmatrix}$

$$= 1\begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & 5 \\ 3 & 1 \end{vmatrix}$$

$$= 1[5 - 1] - 1[1 - 3] + 3[1 - 15]$$

$$= 4 + 2 - 42$$

$$\boxed{S_3 = -36}$$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$$\lambda^3 - 7\lambda^2 + 0\lambda - (-36) = 0$$

$$\lambda^3 - 7\lambda^2 + 0\lambda + 36 = 0$$

<u>To find Eigen values:</u>

$$\begin{array}{r|rrrr} \lambda = -2 & 1 & -7 & 0 & 36 \\ & 0 & -2 & 18 & -36 \\ \hline & 1 & -9 & 18 & \underline{0} \\ & \lambda^2 & \lambda & const & \end{array}$$

$$\lambda^2 - 9\lambda + 18 = 0$$

$$(\lambda - 3)(\lambda - 6) = 0 \quad \begin{bmatrix} \because sum = -9 \text{ and } product = 18 \\ (-3, -6) \end{bmatrix}$$

$$\lambda = 3, \lambda = 6$$

Eigen values are −2, 3, 6

<u>To find Eigen vector:</u>

<u>Case (i) λ = −2:</u>

$$[A - \lambda I]X = 0$$

$$[A - (-2)I]X = 0 \quad \{\because \lambda = -2\}$$

$$[A + 2I]X = 0$$

$$\begin{bmatrix} 1+2 & 1 & 3 \\ 1 & 5+2 & 1 \\ 3 & 1 & 1+2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Select last two rows

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 7 & & 1 & & 1 & & 7 \\ 1 & & 3 & & 3 & & 1 \end{array}$$

$$\frac{x_1}{21 - 1} = \frac{x_2}{3 - 3} = \frac{x_3}{1 - 21}$$

$$\frac{x_1}{20} = \frac{x_2}{0} = \frac{x_3}{-20}$$

Divided by 20

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

Eigen vector $X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

<u>Case (ii) λ = 3:</u>

$$[A - \lambda I]X = 0$$

$$[A - 3.I]X = 0 \quad \{\because \lambda = 3\}$$

$$\begin{bmatrix} 1-3 & 1 & 3 \\ 1 & 5-3 & 1 \\ 3 & 1 & 1-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Select last two rows

$$\begin{matrix} & x_1 & & & x_2 & & x_3 & \\ 2 & & 1 & & & 1 & & 2 \\ 1 & & -2 & & & 3 & & 1 \end{matrix}$$

$$\frac{x_1}{-4-1} = \frac{x_2}{3+2} = \frac{x_3}{1-6}$$

$$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5}$$

Divided by (−5)

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

Eigen vector $X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

<u>Case (iii) λ = 6:</u>

$$[A - \lambda I]X = 0$$

$$[A - 6.I]X = 0 \quad \{\because \lambda = 6\}$$

$$\begin{bmatrix} 1-6 & 1 & 3 \\ 1 & 5-6 & 1 \\ 3 & 1 & 1-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Select last two rows

x_1		x_2	x_3
-1	1	1	-1
1	-5	3	1

$$\frac{x_1}{5-1} = \frac{x_2}{3+5} = \frac{x_3}{1+3}$$

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4}$$

Divided by 4

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

Eigen vector $X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

12. Find the Eigen values and Eigen vectors of $A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$ **[v]**

[M/J 2013 R2008] [A/M 2015 R2008] [N/D 2016 R2008]

Answer:

Matrix $A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

To find Characteristic equation:

$S_1 = Sum\ of\ diagonal\ elements$

$= 2 + 2 + 2$

$\boxed{S_1 = 6}$

$S_2 = Sum\ of\ minor\ of\ diagonal\ elements$

$$= \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}$$

$$= [4 - 0] + [4 - 1] + [4 - 0]$$

$$= 4 + 3 + 4$$

$$\boxed{S_2 = 11}$$

$$S_3 = |A|$$

$$= \begin{vmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ -1 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 2 \\ -1 & 0 \end{vmatrix}$$

$$= 2[4 - 0] - 0 - [0 + 2]$$

$$= 8 - 2$$

$$\boxed{S_3 = 6}$$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

<u>To find Eigen values:</u>

$$\lambda = 1 \left| \begin{array}{cccc} 1 & -6 & 11 & -6 \\ 0 & 1 & -5 & 6 \\ \hline 1 & -5 & 6 & \underline{|0} \\ \lambda^2 & \lambda & const & \end{array} \right.$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 2)(\lambda - 3) = 0 \quad \begin{bmatrix} \because sum = -5 \text{ and } product = 6 \\ (-2,-3) \end{bmatrix}$$

$$\lambda = 2, \lambda = 3$$

Eigen values are 1, 2, 3

<u>To find Eigen vector:</u>

<u>Case (i) λ = 1:</u>

$$[A - \lambda I]X = 0$$

$$[A - 1I]X = 0 \quad \{\because \lambda = 1\}$$

$$\begin{bmatrix} 2-1 & 0 & -1 \\ 0 & 2-1 & 0 \\ -1 & 0 & 2-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Select last two rows

$$\begin{matrix} & x_1 & & x_2 & & x_3 & \\ 1 & & 0 & & 0 & & 1 \\ 0 & & 1 & & -1 & & 0 \end{matrix}$$

$$\frac{x_1}{1-0} = \frac{x_2}{0-0} = \frac{x_3}{0+1}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{1}$$

Eigen vector $X_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

<u>Case (ii) λ = 2:</u>

$$[A - \lambda I]X = 0$$

$$[A - 2I]X = 0 \quad \{\because \lambda = 2\}$$

$$\begin{bmatrix} 2-2 & 0 & -1 \\ 0 & 2-2 & 0 \\ -1 & 0 & 2-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Select first and last rows

$$\begin{matrix} & x_1 & & x_2 & & x_3 & \\ 0 & & -1 & & 0 & & 0 \\ 0 & & 0 & & -1 & & 0 \end{matrix}$$

$$\frac{x_1}{0-0} = \frac{x_2}{1-0} = \frac{x_3}{0-0}$$

$$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{0}$$

Eigen vector $X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

<u>Case (iii) λ = 3:</u>

$$[A - \lambda I]X = 0$$

$$[A - 3.I]X = 0 \quad \{\because \lambda = 3\}$$

$$\begin{bmatrix} 2-3 & 0 & -1 \\ 0 & 2-3 & 0 \\ -1 & 0 & 2-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Select last two rows

$$\begin{matrix} & x_1 & & x_2 & & x_3 & \\ -1 & & 0 & & 0 & & -1 \\ 0 & & -1 & & -1 & & 0 \end{matrix}$$

$$\frac{x_1}{1-0} = \frac{x_2}{0-0} = \frac{x_3}{0-1}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

Eigen vector $X_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

13. Find the Eigen values and Eigen vectors of $\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ **[v] [A/M 2018 R2008]**

<u>Answer:</u>

Matrix $A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

<u>To find Characteristic equation:</u>

$S_1 = Sum\ of\ diagonal\ elements$

$= 2 + 1 - 1$

$\boxed{S_1 = 2}$

$S_2 = Sum\ of\ minor\ of\ diagonal\ elements$

$$= \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix}$$

$$= [-1-3] + [-2-2] + [2+2]$$

$$= -4 - 4 + 4$$

$\boxed{S_2 = -4}$

$S_3 = |A|$

$$= \begin{vmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}$$

$$= 2[-1-3] + 2[-1-1] + 2[3-1]$$

$$= 2[-4] + 2[-2] + 2[2]$$

$$= -8 - 4 + 4$$

$$\boxed{S_3 = -8}$$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$$\lambda^3 - 2\lambda^2 + (-4)\lambda - (-8) = 0$$

$$\lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$$

<u>To find Eigen values:</u>

$$\lambda = 2 \left| \begin{array}{cccc} 1 & -2 & -4 & 8 \\ 0 & 2 & 0 & -8 \\ \hline 1 & 0 & -4 & \underline{|0} \\ \lambda^2 & \lambda & const & \end{array} \right.$$

$$\lambda^2 + 0\lambda - 4 = 0$$

$$\lambda^2 - 4 = 0$$

$$\lambda^2 = 4$$

$$\lambda = \pm\sqrt{4}$$

$$\lambda = \pm 2$$

$$\lambda = 2, \lambda = -2$$

Eigen values are −2, 2, 2

<u>To find Eigen vector:</u>

<u>Case (i) $\lambda = -2$:</u>

$$[A - \lambda I]X = 0$$

$$[A - (-2)I]X = 0 \quad \{\because \lambda = -2\}$$

$$[A + 2I]X = 0$$

$$\begin{bmatrix} 2+2 & -2 & 2 \\ 1 & 1+2 & 1 \\ 1 & 3 & -1+2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & -2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Select first two rows

$$\begin{matrix} & x_1 & & x_2 & & x_3 & \\ -2 & & 2 & & 4 & & -2 \\ 3 & & 1 & & 1 & & 3 \end{matrix}$$

$$\frac{x_1}{-2-6} = \frac{x_2}{2-4} = \frac{x_3}{12+2}$$

$$\frac{x_1}{-8} = \frac{x_2}{-2} = \frac{x_3}{14}$$

Divided by (−2)

$$\frac{x_1}{4} = \frac{x_2}{1} = \frac{x_3}{-7}$$

Eigen vector $X_1 = \begin{bmatrix} 4 \\ 1 \\ -7 \end{bmatrix}$

<u>Case (ii) λ = 2:</u>

$$[A - \lambda I]X = 0$$

$$[A - 2I]X = 0 \quad \{\because \lambda = 2\}$$

$$\begin{bmatrix} 2-2 & -2 & 2 \\ 1 & 1-2 & 1 \\ 1 & 3 & -1-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & -2 & 2 \\ 1 & -1 & 1 \\ 1 & 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Select first two rows

$$\begin{matrix} & x_1 & & x_2 & & x_3 & \\ -2 & & 2 & & 0 & & -2 \\ -1 & & 1 & & 1 & & -1 \end{matrix}$$

$$\frac{x_1}{-2+2} = \frac{x_2}{2-0} = \frac{x_3}{0+2}$$

$$\frac{x_1}{0} = \frac{x_2}{2} = \frac{x_3}{2}$$

Divided by 2

$$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

Eigen vector $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Select last two rows

$$\begin{matrix} & x_1 & & x_2 & & x_3 & \\ -1 & & 1 & & 1 & & -1 \\ 3 & & -3 & & 1 & & 3 \end{matrix}$$

$$\frac{x_1}{3-3} = \frac{x_2}{1+3} = \frac{x_3}{3+1}$$

$$\frac{x_1}{0} = \frac{x_2}{4} = \frac{x_3}{4}$$

Divided by 4

$$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

Eigen vector $X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

14. Find the Eigen values and Eigen vectors of $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ **[B]**

[Jan 2010 R2008][M/J 2014 R2007] [M/J 2014 R2013] [N/D 2014 R2008] [N/D 2015 R2008]

Answer:

Matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

To find Characteristic equation:

$S_1 = Sum\ of\ diagonal\ elements$

$= -2 + 1 + 0$

$\boxed{S_1 = -1}$

$S_2 = Sum\ of\ minor\ of\ diagonal\ elements$

$= \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$

$= [0 - 12] + [0 - 3] + [-2 - 4]$

$= -12 - 3 - 6$

$\boxed{S_2 = -21}$

$S_3 = |A|$

$= \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$

$= -2\begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} - 2\begin{vmatrix} 2 & -6 \\ -1 & 0 \end{vmatrix} + (-3)\begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix}$

$$= -2[0 - 12] - 2[0 - 6] - 3[-4 + 1]$$

$$= 24 + 12 + 9$$

$$\boxed{S_3 = 45}$$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$$\lambda^3 - (-1)\lambda^2 + (-21)\lambda - 45 = 0$$

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

<u>To find Eigen values:</u>

$$\lambda = -3\left|\begin{array}{cccc} 1 & 1 & -21 & -45 \\ 0 & -3 & 6 & 45 \\ \hline 1 & -2 & -15 & \underline{|0} \\ \lambda^2 & \lambda & const & \end{array}\right.$$

$$\lambda^2 - 2\lambda - 15 = 0$$

$$(\lambda + 3)(\lambda - 5) = 0 \quad \begin{bmatrix} \because sum = -2 \text{ and } product = -15 \\ (3, -5) \end{bmatrix}$$

$$\lambda = -3, \lambda = 5$$

Eigen values are 5, −3, −3

<u>To find Eigen vector:</u>

<u>Case (i) λ = 5:</u>

$$[A - \lambda I]X = 0$$

$$[A - 5I]X = 0 \quad \{\because \lambda = 5\}$$

$$\begin{bmatrix} -2-5 & 2 & -3 \\ 2 & 1-5 & -6 \\ -1 & -2 & 0-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Select last two rows

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ -4 & -6 & & 2 & & -4 \\ -2 & -5 & & -1 & & -2 \end{array}$$

$$\frac{x_1}{20 - 12} = \frac{x_2}{6 + 10} = \frac{x_3}{-4 - 4}$$

$$\frac{x_1}{8} = \frac{x_2}{16} = \frac{x_3}{-8}$$

Divided by 8

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1}$$

Eigen vector $X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

<u>Case (ii) λ = −3:</u>

$$\left[A - \lambda I\right]X = 0$$

$$\left[A - (-3)I\right]X = 0 \quad \{\because \lambda = -3\}$$

$$\left[A + 3I\right]X = 0$$

$$\begin{bmatrix} -2+3 & 2 & -3 \\ 2 & 1+3 & -6 \\ -1 & -2 & 0+3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 1 & 2 & -3 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad \left\{\begin{matrix} R_2 \leftrightarrow R_2 \div 2 \\ R_3 \leftrightarrow -R_3 \end{matrix}\right\}$$

$$x_1 + 2x_2 - 3x_3 = 0 \rightarrow (*)$$

Put $x_1 = 0$ in $(*)$	Put $x_2 = 0$ in $(*)$
$2x_2 - 3x_3 = 0$	$x_1 - 3x_3 = 0$
$2x_2 = 3x_3$	$x_1 = 3x_3$
$\frac{x_2}{3} = \frac{x_3}{2}$	$\frac{x_1}{3} = \frac{x_3}{1}$
Eigen vector $X_2 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$	**Eigen vector** $X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

15. Find the Eigen values and Eigen vectors of $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ **[B]**

[M/J 2010 R2008] [N/D 2010 R2008] [Jan 2012 R2008][Jan 2014 R2013]

<u>Answer:</u>

Matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

To find Characteristic equation:

$S_1 = Sum\ of\ diagonal\ elements$

$= 2 + 3 + 2$

$\boxed{S_1 = 7}$

$S_2 = Sum\ of\ minor\ of\ diagonal\ elements$

$= \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix}$

$= [6 - 2] + [4 - 1] + [6 - 2]$

$= 4 + 3 + 4$

$\boxed{S_2 = 11}$

$S_3 = |A|$

$= \begin{vmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix}$

$= 2\begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}$

$= 2[6 - 2] - 2[2 - 1] + 1[2 - 3]$

$= 8 - 2 - 1$

$\boxed{S_3 = 5}$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

To find Eigen values:

$$\begin{array}{r|rrrr} \lambda = 1 & 1 & -7 & 11 & -5 \\ & 0 & 1 & -6 & 5 \\ \hline & 1 & -6 & 5 & \lfloor 0 \\ & \lambda^2 & \lambda & const & \end{array}$$

$\lambda^2 - 6\lambda + 5 = 0$

$(\lambda - 1)(\lambda - 5) = 0 \quad \begin{bmatrix} \because sum = -6\ and\ product = 5 \\ (-1,-5) \end{bmatrix}$

$\lambda = 1, \lambda = 5$

Eigen values are 5, 1, 1

To find Eigen vector:

Case (i) λ = 5:

$$[A - \lambda I]X = 0$$

$$[A - 5I]X = 0 \quad \{\because \lambda = 5\}$$

$$\begin{bmatrix} 2-5 & 2 & 1 \\ 1 & 3-5 & 1 \\ 1 & 2 & 2-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Select last two rows

$$\begin{matrix} & x_1 & & x_2 & & x_3 & \\ -2 & & 1 & & 1 & & -2 \\ 2 & & -3 & & 1 & & 2 \end{matrix}$$

$$\frac{x_1}{6-2} = \frac{x_2}{1+3} = \frac{x_3}{2+2}$$

$$\frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4}$$

Divided by 4

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Eigen vector $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Case (ii) λ = 1:

$$[A - \lambda I]X = 0$$

$$[A - 1.I]X = 0 \quad \{\because \lambda = 1\}$$

$$\begin{bmatrix} 2-1 & 2 & 1 \\ 1 & 3-1 & 1 \\ 1 & 2 & 2-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$x_1 + 2x_2 + x_3 = 0 \rightarrow (*)$$

Put $x_1 = 0$ **in** $(*)$	**Put** $x_2 = 0$ **in** $(*)$
$2x_2 + x_3 = 0$	$x_1 + x_3 = 0$
$2x_2 = -x_3$	$x_1 = -x_3$
$\frac{x_2}{-1} = \frac{x_3}{2}$	$\frac{x_1}{-1} = \frac{x_3}{1}$

Eigen vector $X_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$	Eigen vector $X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

16. Find the Eigen values and Eigen vectors of $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ **[B]**

[A/M 2009 Tirunelveli] [A/M 2015 R2013] [M/J 2016 R2008]

Answer:

Matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

To find Characteristic equation:

S_1 = ***Sum of diagonal elements***

$= 6+3+3$

$\boxed{S_1 = 12}$

S_2 = ***Sum of minor of diagonal elements***

$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$

$= [9-1]+[18-4]+[18-4]$

$= 8+14+14$

$\boxed{S_2 = 36}$

$S_3 = |A|$

$= \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$

$= 6\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} - (-2)\begin{vmatrix} -2 & -1 \\ 2 & 3 \end{vmatrix} + 2\begin{vmatrix} -2 & 3 \\ 2 & -1 \end{vmatrix}$

$= 6[9-1]+2[-6+2]+2[2-6]$

$= 6[8]+2[-4]+2[-4]$

$= 48-8-8$

$\boxed{S_3 = 32}$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

To find Eigen values:

$$\lambda = 2\left|\begin{array}{cccc} 1 & -12 & 36 & -32 \\ 0 & 2 & -20 & 32 \\ \hline 1 & -10 & 16 & \underline{|0} \end{array}\right.$$

$$\lambda^2 \quad \lambda \quad const$$

$$\lambda^2 - 10\lambda + 16 = 0$$

$$(\lambda - 2)(\lambda - 8) = 0 \quad \begin{bmatrix} \because sum = -10 \; and \;\; product = 16 \\ (-2, -8) \end{bmatrix}$$

$$\lambda = 2, \lambda = 8$$

Eigen values are 8, 2, 2

To find Eigen vector:

Case (i) λ = 8:

$$[A - \lambda I] X = 0$$

$$[A - 8.I] X = 0 \quad \{\because \lambda = 8\}$$

$$\begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Select last two rows

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ -5 & -1 & -2 & -5 \\ -1 & -5 & 2 & -1 \end{array}$$

$$\frac{x_1}{25-1} = \frac{x_2}{-2-10} = \frac{x_3}{2+10}$$

$$\frac{x_1}{24} = \frac{x_2}{-12} = \frac{x_3}{12}$$

Divided by 12

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$

Eigen vector $X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Case (ii) λ = 2:

$$\left[A-\lambda I\right]X=0$$

$$\left[A-2.I\right]X=0 \quad \{\because \lambda=2\}$$

$$\begin{bmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

$$\begin{bmatrix} 2 & -1 & 1 \\ 2 & -1 & 1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0 \left\{\begin{matrix} R_1 \leftrightarrow \frac{R_1}{2} \\ R_2 \leftrightarrow -R_2 \end{matrix}\right\}$$

$$2x_1-x_2+x_3=0 \rightarrow (*)$$

Put $x_1=0\ in\ (*)$ $-x_2+x_3=0$ $-x_2=-x_3$ $x_2=x_3$ $\frac{x_2}{1}=\frac{x_3}{1}$ **Eigen vector** $X_2=\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$	**Put** $x_2=0\ in\ (*)$ $2x_1+x_3=0$ $2x_1=-x_3$ $\frac{x_1}{-1}=\frac{x_3}{2}$ **Eigen vector** $X_3=\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

17. Reduce $8x^2+7y^2+3z^2-12xy+4xz-8yz$ **into canonical form by orthogonal reduction. [B]**

(Or)

Reduce the Quadratic form $8x_1^2+7x_2^2+3x_3^2-8x_2x_3+4x_1x_3-12x_1x_2$ **to canonical form through an orthogonal transformation and find the rank, index, signature and nature of the quadratic form [v ex]**

[A/M 2009 R2004] [N/D 2011 R2008] [N/D 2015 R2008]

Answer:

To find Matrix:

Given Equation, $8x^2+7y^2+3z^2-12xy+4xz-8yz$

$$Matrix = \begin{bmatrix} coeff\ x^2 & \frac{1}{2}coeff\ xy & \frac{1}{2}coeff\ xz \\ \frac{1}{2}coeff\ xy & coeff\ y^2 & \frac{1}{2}coeff\ yz \\ \frac{1}{2}coeff\ xz & \frac{1}{2}coeff\ yz & coeff\ z^2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & \frac{1}{2}(-12) & \frac{1}{2}(4) \\ \frac{1}{2}(-12) & 7 & \frac{1}{2}(-8) \\ \frac{1}{2}(4) & \frac{1}{2}(-8) & 3 \end{bmatrix}$$

Matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

To find Characteristic equation:

$S_1 = Sum\ of\ diagonal\ elements$

$= 8+7+3$

$\boxed{S_1 = 18}$

$S_2 = Sum\ of\ minor\ of\ diagonal\ elements$

$$= \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$= [21-16]+[24-4]+[56-36]$

$= 5+20+20$

$\boxed{S_2 = 45}$

$S_3 = |A|$

$$= \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$$= 8\begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} - (-6)\begin{vmatrix} -6 & -4 \\ 2 & 3 \end{vmatrix} + 2\begin{vmatrix} -6 & 7 \\ 2 & -4 \end{vmatrix}$$

$$= 8[21 - 16] + 6[-18 + 8] + 2[24 - 14]$$

$$= 8[5] + 6[-10] + 2[10]$$

$$= 40 - 60 + 20$$

$$\boxed{S_3 = 0}$$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$$\lambda^3 - 18\lambda^2 + 45\lambda - 0 = 0$$

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

<u>To find Eigen values:</u>

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\lambda\left(\lambda^2 - 18\lambda + 45\right) = 0$$

$$\lambda = 0 \ \ or \ \lambda^2 - 18\lambda + 45 = 0$$

$$(\lambda - 3)(\lambda - 15) = 0 \ \begin{bmatrix} \because sum = -18 \ and \ \ product = 45 \\ (-3, -15) \end{bmatrix}$$

$$\lambda = 3, \lambda = 15$$

Eigen values are 0, 3, 15

<u>To find Eigen vector:</u>

<u>Case (i) λ = 0:</u>

$$[A - \lambda I]X = 0$$

$$[A - 0.I]X = 0 \quad \{\because \lambda = 0\}$$

$$AX = 0$$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Select last two rows

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ 7 & -4 & -6 & 7 \\ -4 & 3 & 2 & -4 \end{array}$$

$$\frac{x_1}{21 - 16} = \frac{x_2}{-8 + 18} = \frac{x_3}{24 - 14}$$

$$\frac{x_1}{5} = \frac{x_2}{10} = \frac{x_3}{10}$$

Divided by 5

$$\frac{x_1}{1}=\frac{x_2}{2}=\frac{x_3}{2}$$

Eigen vector $X_1=\begin{bmatrix}1\\2\\2\end{bmatrix}$

<u>Case (ii) λ = 3:</u>

$$\left[A-\lambda I\right]X=0$$

$$\left[A-3I\right]X=0 \quad \{\because \lambda=3\}$$

$$\begin{bmatrix}8-3 & -6 & 2\\ -6 & 7-3 & -4\\ 2 & -4 & 3-3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$$

$$\begin{bmatrix}5 & -6 & 2\\ -6 & 4 & -4\\ 2 & -4 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0$$

Select last two rows

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 4 & -4 & & -6 & 4 & \\ -4 & 0 & & 2 & -4 & \end{array}$$

$$\frac{x_1}{0-16}=\frac{x_2}{-8-0}=\frac{x_3}{24-8}$$

$$\frac{x_1}{-16}=\frac{x_2}{-8}=\frac{x_3}{16}$$

Divided by (−8)

$$\frac{x_1}{2}=\frac{x_2}{1}=\frac{x_3}{-2}$$

Eigen vector $X_2=\begin{bmatrix}2\\1\\-2\end{bmatrix}$

<u>Case (iii) λ = 15:</u>

$$\left[A-\lambda I\right]X=0$$

$$\left[A-15I\right]X=0 \quad \{\because \lambda=15\}$$

$$\begin{bmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

Select last two rows

$$\begin{matrix} & x_1 & x_2 & x_3 & \\ -8 & -4 & -6 & -8 \\ -4 & -12 & 2 & -4 \end{matrix}$$

$$\frac{x_1}{96-16}=\frac{x_2}{-8-72}=\frac{x_3}{24+16}$$

$$\frac{x_1}{80}=\frac{x_2}{-80}=\frac{x_3}{40}$$

Divided by 40

$$\frac{x_1}{2}=\frac{x_2}{-2}=\frac{x_3}{1}$$

Eigen vector $X_3=\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

To check orthogonal property:

$$X_1^T X_2=\begin{bmatrix} 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}=\begin{bmatrix} 2+2-4 \end{bmatrix}=\begin{bmatrix} 0 \end{bmatrix}=0$$

$$X_2^T X_3=\begin{bmatrix} 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}=\begin{bmatrix} 4-2-2 \end{bmatrix}=\begin{bmatrix} 0 \end{bmatrix}=0$$

$$X_3^T X_1=\begin{bmatrix} 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}=\begin{bmatrix} 2-4+2 \end{bmatrix}=\begin{bmatrix} 0 \end{bmatrix}=0$$

X_1, X_2, X_3 **are orthogonal in pairs.**

To find Model Matrix N:

Eigen vector	$\sqrt{x_1^2+x_2^2+x_3^2}$	Normalized Eigen vector

$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$	$\sqrt{1^2 + 2^2 + 2^2}$ $= \sqrt{9}$ $= 3$	$\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$
$X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$	$\sqrt{2^2 + 1^2 + (-2)^2}$ $= \sqrt{9}$ $= 3$	$\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{-2}{3} \end{bmatrix}$
$X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$	$\sqrt{2^2 + (-2)^2 + 1^2}$ $= \sqrt{9}$ $= 3$	$\begin{bmatrix} \frac{2}{3} \\ \frac{-2}{3} \\ \frac{1}{3} \end{bmatrix}$

$$N = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

To find Diagonal Matrix D:

$$D = N^T A N$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 8-12+4 & 16-6-4 & 16+12+2 \\ -6+14-8 & -12+7+8 & -12-14-4 \\ 2-8+6 & 4-4-6 & 4+8+3 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 6 & 30 \\ 0 & 3 & -30 \\ 0 & -6 & 15 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 0+0+0 & 6+6-12 & 30-60+30 \\ 0+0+0 & 12+3+12 & 60-30-30 \\ 0+0+0 & 12-6-6 & 60+60+15 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 135 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

To find canonical form:

$$Y^T DY = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 0y_1^2 + 3y_2^2 + 15y_3^2$$

$$= 3y_2^2 + 15y_3^2$$

To find Rank, index, signature and nature:

Rank = Number of square terms.

=2

Index = Number of positive square terms.

= 2

Signature = Number of positive square terms – Number of negative square terms.

= 2−0

= 2

Nature = positive semi definite.

18. Reduce the Quadratic form $x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$ **to canonical form through an orthogonal transformation and find the rank, index, signature and nature of the quadratic form [v]**

[A/M 2008 R2008] [M/J 2009 R2008]

Answer:

To find Matrix:

Given Equation, $x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$

$$Matrix = \begin{bmatrix} coeff\ x_1^2 & \frac{1}{2}coeff\ x_1x_2 & \frac{1}{2}coeff\ x_1x_3 \\ \frac{1}{2}coeff\ x_1x_2 & coeff\ x_2^2 & \frac{1}{2}coeff\ x_2x_3 \\ \frac{1}{2}coeff\ x_1x_3 & \frac{1}{2}coeff\ x_2x_3 & coeff\ x_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{1}{2}(-2) & \frac{1}{2}(0) \\ \frac{1}{2}(-2) & 2 & \frac{1}{2}(2) \\ \frac{1}{2}(0) & \frac{1}{2}(2) & 1 \end{bmatrix}$$

Matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

To find Characteristic equation:

S_1 = *Sum of diagonal elements*

$=1+2+1$

$\boxed{S_1 = 4}$

S_2 = *Sum of minor of diagonal elements*

$$= \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}$$

$=[2-1]+[1-0]+[2-1]$

$=1+1+1$

$\boxed{S_2 = 3}$

$S_3 = |A|$

$$= \begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$=1\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} - (-1)\begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix}$$

$=[2-1]+[-1-0]+0$

$=1-1$

$\boxed{S_3 = 0}$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$$\lambda^3 - 4\lambda^2 + 3\lambda - 0 = 0$$

$$\lambda^3 - 4\lambda^2 + 3\lambda = 0$$

<u>To find Eigen values:</u>

$$\lambda^3 - 4\lambda^2 + 3\lambda = 0$$

$$\lambda\left(\lambda^2 - 4\lambda + 3\right) = 0$$

$$\lambda = 0 \ \ or \ \lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 1)(\lambda - 3) = 0 \quad \begin{bmatrix} \because sum = -4 \ and \ \ product = 3 \\ (-1, -3) \end{bmatrix}$$

$$\lambda = 1, \lambda = 3$$

Eigen values are 0, 1, 3

<u>To find Eigen vector:</u>

<u>Case (i) λ = 0:</u>

$$[A - \lambda I]X = 0$$

$$[A - 0.I]X = 0 \quad \{\because \lambda = 0\}$$

$$AX = 0$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Select last two rows

$$\begin{matrix} & x_1 & & x_2 & & x_3 & \\ 2 & & 1 & & -1 & & 2 \\ 1 & & 1 & & 0 & & 1 \end{matrix}$$

$$\frac{x_1}{2-1} = \frac{x_2}{0+1} = \frac{x_3}{-1-0}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{-1}$$

Eigen vector $X_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

<u>Case (ii) λ = 1:</u>

$$[A - \lambda I]X = 0$$

$$[A - 1.I]X = 0 \quad \{\because \lambda = 1\}$$

$$\begin{bmatrix} 1-1 & -1 & 0 \\ -1 & 2-1 & 1 \\ 0 & 1 & 1-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Select last two rows

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 1 & 1 & & -1 & & 1 \\ 1 & 0 & & 0 & & 1 \end{array}$$

$$\frac{x_1}{0-1} = \frac{x_2}{0-0} = \frac{x_3}{-1-0}$$

$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

Divided by (−1)

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{1}$$

Eigen vector $X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

<u>Case (iii) λ = 3:</u>

$$\left[A - \lambda I\right] X = 0$$

$$\left[A - 3.I\right] X = 0 \quad \{\because \lambda = 3\}$$

$$\begin{bmatrix} 1-3 & -1 & 0 \\ -1 & 2-3 & 1 \\ 0 & 1 & 1-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Select last two rows

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ -1 & 1 & & -1 & & -1 \\ 1 & -2 & & 0 & & 1 \end{array}$$

$$\frac{x_1}{2-1} = \frac{x_2}{0-2} = \frac{x_3}{-1-0}$$

$$\frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{-1}$$

Eigen vector $X_3 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$

<u>To check orthogonal property:</u>

$$X_1^T X_2 = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+0-1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} = 0$$

$$X_2^T X_3 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1+0-1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} = 0$$

$$X_3^T X_1 = \begin{bmatrix} 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1-2+1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} = 0$$

X_1, X_2, X_3 **are orthogonal in pairs.**

<u>To find Model Matrix N:</u>

Eigen vector	$\sqrt{x_1^2 + x_2^2 + x_3^2}$	**Normalized Eigen vector**
$X_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$	$\sqrt{1^2 + 1^2 + (-1)^2}$ $= \sqrt{3}$	$\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{bmatrix}$
$X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$	$\sqrt{1^2 + 0^2 + 1^2}$ $= \sqrt{2}$	$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$
$X_3 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$	$\sqrt{1^2 + (-2)^2 + (-1)^2}$ $= \sqrt{6}$	$\begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \end{bmatrix}$

$$N = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \end{bmatrix}$$

<u>To find Diagonal Matrix D:</u>

$D = N^T AN$

$$=\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}}+0 & \frac{1}{\sqrt{2}}+0+0 & \frac{1}{\sqrt{6}}+\frac{2}{\sqrt{6}}+0 \\ \frac{-1}{\sqrt{3}}+\frac{2}{\sqrt{3}}-\frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}}+0+\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}}-\frac{4}{\sqrt{6}}-\frac{1}{\sqrt{6}} \\ 0+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}} & 0+0+\frac{1}{\sqrt{2}} & 0-\frac{2}{\sqrt{6}}-\frac{1}{\sqrt{6}} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{6}} \\ 0 & 0 & \frac{-6}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-3}{\sqrt{6}} \end{bmatrix}$$

$$=\begin{bmatrix} 0+0+0 & \frac{1}{\sqrt{6}}+0-\frac{1}{\sqrt{6}} & \frac{3}{\sqrt{18}}-\frac{6}{\sqrt{18}}+\frac{3}{\sqrt{18}} \\ 0+0+0 & \frac{1}{\sqrt{4}}+0+\frac{1}{\sqrt{4}} & \frac{3}{\sqrt{12}}+0-\frac{3}{\sqrt{12}} \\ 0+0+0 & \frac{1}{\sqrt{12}}+0-\frac{1}{\sqrt{12}} & \frac{3}{\sqrt{36}}+\frac{12}{\sqrt{36}}+\frac{3}{\sqrt{36}} \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{2}{\sqrt{4}} & 0 \\ 0 & 0 & \frac{18}{\sqrt{36}} \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{2}{2} & 0 \\ 0 & 0 & \frac{18}{6} \end{bmatrix}$$

$$D=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

To find canonical form:

$$Y^T DY = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 0y_1^2 + 1y_2^2 + 3y_3^2$$

$$= y_2^2 + 3y_3^2$$

<u>To find Rank, index, signature and nature:</u>

Rank = Number of square terms.

=2

Index = Number of positive square terms.

= 2

Signature = Number of positive square terms – Number of negative square terms.

= 2−0

= 2

Nature = positive semi definite.

19. Reduce the Quadratic form $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy$ to the canonical form and specify the matrix of transformation. [G] [Jan 2011 R2010] [M/J 2013 R2008] [N/D 2014 R2013]

<u>Answer:</u>

<u>To find Matrix:</u>

Given Equation, $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy$

$$Matrix = \begin{bmatrix} coeff\ x^2 & \frac{1}{2}coeff\ xy & \frac{1}{2}coeff\ xz \\ \frac{1}{2}coeff\ xy & coeff\ y^2 & \frac{1}{2}coeff\ yz \\ \frac{1}{2}coeff\ xz & \frac{1}{2}coeff\ yz & coeff\ z^2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & \frac{1}{2}(-2) & \frac{1}{2}(2) \\ \frac{1}{2}(-2) & 5 & \frac{1}{2}(-2) \\ \frac{1}{2}(2) & \frac{1}{2}(-2) & 3 \end{bmatrix}$$

Matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

To find Characteristic equation:

S_1 = *Sum of diagonal elements*

$= 3 + 5 + 3$

$\boxed{S_1 = 11}$

S_2 = *Sum of minor of diagonal elements*

$= \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$

$= [15 - 1] + [9 - 1] + [15 - 1]$

$= 14 + 8 + 14$

$\boxed{S_2 = 36}$

$S_3 = |A|$

$= \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix}$

$= 3\begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} -1 & 5 \\ 1 & -1 \end{vmatrix}$

$= 3[15 - 1] + [-3 + 1] + [1 - 5]$

$= 3[14] + [-2] + [-4]$

$= 42 - 2 - 4$

$\boxed{S_3 = 36}$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

To find Eigen values:

$$\begin{array}{r|rrrr} \lambda = 2 & 1 & -11 & 36 & -36 \\ & 0 & 2 & -18 & 36 \\ \hline & 1 & -9 & 18 & \underline{|0} \\ & \lambda^2 & \lambda & const & \end{array}$$

$\lambda^2 - 9\lambda + 18 = 0$

$(\lambda - 3)(\lambda - 6) = 0 \quad \begin{bmatrix} \because sum = -9 \text{ and } product = 18 \\ (-3, -6) \end{bmatrix}$

$$\lambda = 3, \lambda = 6$$

Eigen values are 2, 3, 6

To find Eigen vector:

Case (i) λ = 2:

$$[A - \lambda I]X = 0$$

$$[A - 2.I]X = 0 \quad \{\because \lambda = 2\}$$

$$\begin{bmatrix} 3-2 & -1 & 1 \\ -1 & 5-2 & -1 \\ 1 & -1 & 3-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Select last two rows

$$\begin{matrix} & x_1 & & x_2 & & x_3 & \\ 3 & & -1 & & -1 & & 3 \\ -1 & & 1 & & 1 & & -1 \end{matrix}$$

$$\frac{x_1}{3-1} = \frac{x_2}{-1+1} = \frac{x_3}{1-3}$$

$$\frac{x_1}{2} = \frac{x_2}{0} = \frac{x_3}{-2}$$

Divided by 2

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

Eigen vector $X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Case (ii) λ = 3:

$$[A - \lambda I]X = 0$$

$$[A - 3.I]X = 0 \quad \{\because \lambda = 3\}$$

$$\begin{bmatrix} 3-3 & -1 & 1 \\ -1 & 5-3 & -1 \\ 1 & -1 & 3-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Select last two rows

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 2 & -1 & -1 & 2 \\ -1 & 0 & 1 & -1 \end{array}$$

$$\frac{x_1}{0-1} = \frac{x_2}{-1-0} = \frac{x_3}{1-2}$$

$$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

Divided by (−1)

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Eigen vector $X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Case (iii) λ = 6:

$$\left[A - \lambda I\right] X = 0$$

$$\left[A - 6.I\right] X = 0 \quad \{\because \lambda = 6\}$$

$$\begin{bmatrix} 3-6 & -1 & 1 \\ -1 & 5-6 & -1 \\ 1 & -1 & 3-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Select last two rows

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ -1 & -1 & -1 & -1 \\ -1 & -3 & 1 & -1 \end{array}$$

$$\frac{x_1}{3-1} = \frac{x_2}{-1-3} = \frac{x_3}{1+1}$$

$$\frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{2}$$

Divided by 2

$$\frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{1}$$

Eigen vector $X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

To check orthogonal property:

$$X_1^T X_2 = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \left[1+0-1\right] = [0] = 0$$

$$X_2^T X_3 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-2+1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} = 0$$

$$X_3^T X_1 = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1+0-1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} = 0$$

X_1, X_2, X_3 **are orthogonal in pairs.**

<u>To find Model Matrix N:</u>

Eigen vector	$\sqrt{x_1^2 + x_2^2 + x_3^2}$	**Normalized Eigen vector**
$X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$	$\sqrt{(1)^2 + (0)^2 + (-1)^2}$ $= \sqrt{2}$	$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$
$X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\sqrt{(1)^2 + (1)^2 + (1)^2}$ $= \sqrt{3}$	$\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$
$X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$	$\sqrt{(1)^2 + (-2)^2 + (1)^2}$ $= \sqrt{6}$	$\begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$

$$N=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}}\\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\end{bmatrix}$$

To find Diagonal Matrix D:

$$D=N^T AN$$

$$=\begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}}\end{bmatrix}\begin{bmatrix}3 & -1 & 1\\ -1 & 5 & -1\\ 1 & -1 & 3\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}}\\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}}\end{bmatrix}\begin{bmatrix}\frac{3}{\sqrt{2}}+0-\frac{1}{\sqrt{2}} & \frac{3}{\sqrt{3}}-\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}} & \frac{3}{\sqrt{6}}+\frac{2}{\sqrt{6}}+\frac{1}{\sqrt{6}}\\ -\frac{1}{\sqrt{2}}+0+\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}}+\frac{5}{\sqrt{3}}-\frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}}-\frac{10}{\sqrt{6}}-\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{2}}+0-\frac{3}{\sqrt{2}} & \frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}}+\frac{3}{\sqrt{3}} & \frac{1}{\sqrt{6}}+\frac{2}{\sqrt{6}}+\frac{3}{\sqrt{6}}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}}\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}}\\ 0 & \frac{3}{\sqrt{3}} & \frac{-12}{\sqrt{6}}\\ \frac{-2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{6}{\sqrt{6}}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{2}{\sqrt{4}}+0+\frac{2}{\sqrt{4}} & \frac{3}{\sqrt{6}}+0-\frac{3}{\sqrt{6}} & \frac{6}{\sqrt{12}}+0-\frac{6}{\sqrt{12}}\\ \frac{2}{\sqrt{6}}+0-\frac{2}{\sqrt{6}} & \frac{3}{\sqrt{9}}+\frac{3}{\sqrt{9}}+\frac{3}{\sqrt{9}} & \frac{6}{\sqrt{18}}-\frac{12}{\sqrt{18}}+\frac{6}{\sqrt{18}}\\ \frac{2}{\sqrt{12}}+0-\frac{2}{\sqrt{12}} & \frac{3}{\sqrt{18}}-\frac{6}{\sqrt{18}}+\frac{3}{\sqrt{18}} & \frac{6}{\sqrt{36}}+\frac{24}{\sqrt{36}}+\frac{6}{\sqrt{36}}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{4}{\sqrt{4}} & 0 & 0\\ 0 & \frac{9}{\sqrt{9}} & 0\\ 0 & 0 & \frac{36}{\sqrt{36}}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{4}{2} & 0 & 0\\ 0 & \frac{9}{3} & 0\\ 0 & 0 & \frac{36}{6}\end{bmatrix}$$

$$D=\begin{bmatrix}2&0&0\\0&3&0\\0&0&6\end{bmatrix}$$

To find canonical form:

$$Y^T DY=\begin{bmatrix}y_1 & y_2 & y_3\end{bmatrix}\begin{bmatrix}2&0&0\\0&3&0\\0&0&6\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$$

$$=2y_1^2+3y_2^2+6y_3^2$$

To find Rank, index, signature and nature:

Rank = Number of square terms.

=3

Index = Number of positive square terms.

= 3

Signature = Number of positive square terms – Number of negative square terms.

= 3−0

= 3

Nature = positive definite.

20. Reduce the Quadratic form $2x_1^2+x_2^2+x_3^2+2x_1x_2-2x_1x_3-4x_2x_3$ to canonical form through an orthogonal transformation and find the rank, index, signature and nature of the quadratic form [v]

[N/D 2010 R2008] [M/J 2012 R2008] [A/M 2015 R2008] [N/D 2016 R2008]

Answer:

To find Matrix:

Given Equation, $2x_1^2+x_2^2+x_3^2+2x_1x_2-2x_1x_3-4x_2x_3$

$$Matrix=\begin{bmatrix}coeff\ x_1^2 & \frac{1}{2}coeff\ x_1x_2 & \frac{1}{2}coeff\ x_1x_3\\ \frac{1}{2}coeff\ x_1x_2 & coeff\ x_2^2 & \frac{1}{2}coeff\ x_2x_3\\ \frac{1}{2}coeff\ x_1x_3 & \frac{1}{2}coeff\ x_2x_3 & coeff\ x_3^2\end{bmatrix}$$

$$=\begin{bmatrix}2 & \frac{1}{2}(2) & \frac{1}{2}(-2)\\ \frac{1}{2}(2) & 1 & \frac{1}{2}(-4)\\ \frac{1}{2}(-2) & \frac{1}{2}(-4) & 1\end{bmatrix}$$

Matrix $A=\begin{bmatrix}2&1&-1\\1&1&-2\\-1&-2&1\end{bmatrix}$

To find Characteristic equation:

$S_1 = Sum\ of\ diagonal\ elements$

$= 2 + 1 + 1$

$\boxed{S_1 = 4}$

$S_2 = Sum\ of\ minor\ of\ diagonal\ elements$

$= \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$

$= [1 - 4] + [2 - 1] + [2 - 1]$

$= -3 + 1 + 1$

$\boxed{S_2 = -1}$

$S_3 = |A|$

$= \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{vmatrix}$

$= 2\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix}$

$= 2[1 - 4] - [1 - 2] - [-2 + 1]$

$= 2[-3] - [-1] - [-1]$

$= -6 + 1 + 1$

$\boxed{S_3 = -4}$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$$\lambda^3 - 4\lambda^2 + (-1)\lambda - (-4) = 0$$

$$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

<u>To find Eigen values:</u>

$$\begin{array}{r|rrrr} \lambda = 1 & 1 & -4 & -1 & 4 \\ & 0 & 1 & -3 & -4 \\ \hline & 1 & -3 & -4 & |\underline{0} \\ & \lambda^2 & \lambda & const & \end{array}$$

$\lambda^2 - 3\lambda - 4 = 0$

$(\lambda + 1)(\lambda - 4) = 0 \quad \left[\begin{array}{l} \because sum = -3\ and\ product = -4 \\ (1, -4) \end{array}\right]$

$\lambda = -1, \lambda = 4$

Eigen values are 1, −1, 4

<u>To find Eigen vector:</u>

<u>Case (i) λ = 1:</u>

$[A - \lambda I]X = 0$

$$\left[A - 1.I\right]X = 0 \quad \{\because \lambda = 1\}$$

$$\begin{bmatrix} 2-1 & 1 & -1 \\ 1 & 1-1 & -2 \\ -1 & -2 & 1-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & -2 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Select last two rows

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 0 & -2 & 1 & 0 \\ -2 & 0 & -1 & -2 \end{array}$$

$$\frac{x_1}{0-4} = \frac{x_2}{2-0} = \frac{x_3}{-2-0}$$

$$\frac{x_1}{-4} = \frac{x_2}{2} = \frac{x_3}{-2}$$

Divided by (−2)

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$

Eigen vector $X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

<u>Case (ii) λ = −1:</u>

$$\left[A - \lambda I\right]X = 0$$

$$\left[A - (-1).I\right]X = 0 \quad \{\because \lambda = -1\}$$

$$\left[A + 1.I\right]X = 0$$

$$\begin{bmatrix} 2+1 & 1 & -1 \\ 1 & 1+1 & -2 \\ -1 & -2 & 1+1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & -2 \\ -1 & -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & -2 \\ 1 & 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \{R_3 \leftrightarrow -R_3\}$$

Select first two rows

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 1 & -1 & 3 & 1 \\ 2 & -2 & 1 & 2 \end{array}$$

$$\frac{x_1}{-2+2} = \frac{x_2}{-1+6} = \frac{x_3}{6-1}$$

$$\frac{x_1}{0} = \frac{x_2}{5} = \frac{x_3}{5}$$

Divided by 5

$$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

Eigen vector $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Case (iii) λ = 4:

$$\left[A - \lambda I\right]X = 0$$

$$\left[A - 4.I\right]X = 0 \quad \{\because \lambda = 4\}$$

$$\begin{bmatrix} 2-4 & 1 & -1 \\ 1 & 1-4 & -2 \\ -1 & -2 & 1-4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & 1 & -1 \\ 1 & -3 & -2 \\ -1 & -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Select last two rows

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ -3 & -2 & 1 & -3 \\ -2 & -3 & -1 & -2 \end{array}$$

$$\frac{x_1}{9-4} = \frac{x_2}{2+3} = \frac{x_3}{-2-3}$$

$$\frac{x_1}{5} = \frac{x_2}{5} = \frac{x_3}{-5}$$

Divided by 5

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{-1}$$

Eigen vector $X_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

To check orthogonal property:

$$X_1^T X_2 = \begin{bmatrix} 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 - 1 + 1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} = 0$$

$$X_2^T X_3 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 + 1 - 1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} = 0$$

$$X_3^T X_1 = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 - 1 - 1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} = 0$$

X_1, X_2, X_3 **are orthogonal in pairs.**

To find Model Matrix N:

Eigen vector	$\sqrt{x_1^2 + x_2^2 + x_3^2}$	Normalized Eigen vector
$X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$	$\sqrt{(2)^2 + (-1)^2 + (1)^2}$ $= \sqrt{6}$	$\begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$
$X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$	$\sqrt{(0)^2 + (1)^2 + (1)^2}$ $= \sqrt{2}$	$\begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$
$X_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$	$\sqrt{(1)^2 + (1)^2 + (-1)^2}$ $= \sqrt{3}$	$\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{bmatrix}$

$$N = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

To find Diagonal Matrix D:

$$D = N^T AN$$

$$= \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{4}{\sqrt{6}} - \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}} & 0 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} - \frac{1}{\sqrt{6}} - \frac{2}{\sqrt{6}} & 0 + \frac{1}{\sqrt{2}} - \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} \\ \frac{-2}{\sqrt{6}} + \frac{2}{\sqrt{6}} + \frac{1}{\sqrt{6}} & 0 - \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} - \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{4}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{4}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{-4}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{\sqrt{36}} + \frac{1}{\sqrt{36}} + \frac{1}{\sqrt{36}} & 0 + \frac{1}{\sqrt{12}} - \frac{1}{\sqrt{12}} & \frac{8}{\sqrt{18}} - \frac{4}{\sqrt{18}} - \frac{4}{\sqrt{18}} \\ 0 - \frac{1}{\sqrt{12}} + \frac{1}{\sqrt{12}} & 0 - \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{4}} & 0 + \frac{4}{\sqrt{6}} - \frac{4}{\sqrt{6}} \\ \frac{2}{\sqrt{18}} - \frac{1}{\sqrt{18}} - \frac{1}{\sqrt{18}} & 0 - \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}} & \frac{4}{\sqrt{9}} + \frac{4}{\sqrt{9}} + \frac{4}{\sqrt{9}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{6}{\sqrt{36}} & 0 & 0 \\ 0 & -\frac{2}{\sqrt{4}} & 0 \\ 0 & 0 & \frac{12}{\sqrt{9}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{6}{6} & 0 & 0 \\ 0 & -\frac{2}{2} & 0 \\ 0 & 0 & \frac{12}{3} \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

<u>To find canonical form:</u>

$$Y^T DY = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 1y_1^2 - 1y_2^2 + 4y_3^2$$

<u>To find Rank, index, signature and nature:</u>

Rank = Number of square terms.

=3

Index = Number of positive square terms.

= 2

Signature = Number of positive square terms – Number of negative square terms.

= 2−1

= 1

Nature = Indefinite.

21. Reduce the quadratic form $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$ **to its canonical form through an orthogonal transformation and find the rank, index, signature and nature of the quadratic form [v ex]**

(Or)

Reduce $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$ **into canonical form by an orthogonal reduction and find the rank, index, signature and nature of the quadratic form [B]**

[Jan 2014 R2013] [N/D 2015 R2013] [M/J 2016 R2013, R2008] [N/D 2016 R2013]

<u>Answer:</u>

<u>To find Matrix:</u>

Given Equation, $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$

$$Matrix = \begin{bmatrix} coeff\ x_1^2 & \frac{1}{2} coeff\ x_1x_2 & \frac{1}{2} coeff\ x_1x_3 \\ \frac{1}{2} coeff\ x_1x_2 & coeff\ x_2^2 & \frac{1}{2} coeff\ x_2x_3 \\ \frac{1}{2} coeff\ x_1x_3 & \frac{1}{2} coeff\ x_2x_3 & coeff\ x_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & \frac{1}{2}(-4) & \frac{1}{2}(4) \\ \frac{1}{2}(-4) & 3 & \frac{1}{2}(-2) \\ \frac{1}{2}(4) & \frac{1}{2}(-2) & 3 \end{bmatrix}$$

Matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

To find Characteristic equation:

S_1 = Sum of diagonal elements

$= 6 + 3 + 3$

$\boxed{S_1 = 12}$

S_2 = Sum of minor of diagonal elements

$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$

$= [9-1] + [18-4] + [18-4]$

$= 8 + 14 + 14$

$\boxed{S_2 = 36}$

$S_3 = |A|$

$= \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$

$= 6\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} - (-2)\begin{vmatrix} -2 & -1 \\ 2 & 3 \end{vmatrix} + 2\begin{vmatrix} -2 & 3 \\ 2 & -1 \end{vmatrix}$

$= 6[9-1] + 2[-6+2] + 2[2-6]$

$= 6[8] + 2[-4] + 2[-4]$

$= 48 - 8 - 8$

$\boxed{S_3 = 32}$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

To find Eigen values:

$$\lambda = 2 \left| \begin{array}{cccc} 1 & -12 & 36 & -32 \\ 0 & 2 & -20 & 32 \\ \hline 1 & -10 & 16 & \underline{|0} \\ \lambda^2 & \lambda & const & \end{array} \right.$$

$\lambda^2 - 10\lambda + 16 = 0$

$$(\lambda-2)(\lambda-8)=0 \quad \begin{bmatrix} \because sum=-10 \; and \; product=16 \\ (-2,-8) \end{bmatrix}$$

$$\lambda=2, \lambda=8$$

Eigen values are 8, 2, 2

To find Eigen vector:

Case (i) λ = 8:

$$[A-\lambda I]X=0$$

$$[A-8.I]X=0 \quad \{\because \lambda=8\}$$

$$\begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

Select last two rows

$$\begin{matrix} x_1 & x_2 & x_3 & \\ -5 & -1 & -2 & -5 \\ -1 & -5 & 2 & -1 \end{matrix}$$

$$\frac{x_1}{25-1}=\frac{x_2}{-2-10}=\frac{x_3}{2+10}$$

$$\frac{x_1}{24}=\frac{x_2}{-12}=\frac{x_3}{12}$$

Divided by 12

$$\frac{x_1}{2}=\frac{x_2}{-1}=\frac{x_3}{1}$$

Eigen vector $X_1=\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Case (ii) λ = 2:

$$[A-\lambda I]X=0$$

$$[A-2.I]X=0 \quad \{\because \lambda=2\}$$

$$\begin{bmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

$$\begin{bmatrix} 2 & -1 & 1 \\ 2 & -1 & 1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \left\{ \begin{matrix} R_1 \leftrightarrow \frac{R_1}{2} \\ R_2 \leftrightarrow -R_2 \end{matrix} \right\}$$

$$2x_1 - x_2 + x_3 = 0 \to (*)$$

Put $x_1 = 0\ in\ (*)$	**Put** $x_2 = 0\ in\ (*)$
$-x_2 + x_3 = 0$	$2x_1 + x_3 = 0$
$-x_2 = -x_3$	$2x_1 = -x_3$
$x_2 = x_3$	$\frac{x_1}{-1} = \frac{x_3}{2}$
$\frac{x_2}{1} = \frac{x_3}{1}$	**Eigen vector** $X_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$
Eigen vector $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$	

<u>To check orthogonal property:</u>

$$X_1^T X_2 = \begin{bmatrix} 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 - 1 + 1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} = 0$$

$$X_2^T X_3 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 + 0 + 2 \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix} \neq 0$$

$$X_3^T X_1 = \begin{bmatrix} -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 + 0 + 2 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} = 0$$

X_1, X_2, X_3 **are not orthogonal in pairs.**

Choose $X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ **such that** $X_2^T X_3 = 0$ **and** $X_3^T X_1 = 0$

$$\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \textbf{ and } \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = 0$$

$$\begin{matrix} & x_1 & & x_2 & & x_3 & \\ 1 & & 1 & & 0 & & 1 \\ -1 & & 1 & & 2 & & -1 \end{matrix}$$

$$\frac{x_1}{1+1} = \frac{x_2}{2-0} = \frac{x_3}{0-2}$$

$$\frac{x_1}{2} = \frac{x_2}{2} = \frac{x_3}{-2}$$

Divided by 2

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$$\textbf{Eigen vector } X_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

To find Model Matrix N:

Eigen vector	$\sqrt{x_1^2 + x_2^2 + x_3^2}$	Normalized Eigen vector
$X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$	$\sqrt{(2)^2 + (-1)^2 + (1)^2}$ $= \sqrt{6}$	$\begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$
$X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$	$\sqrt{(0)^2 + (1)^2 + (1)^2}$ $= \sqrt{2}$	$\begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$
$X_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$	$\sqrt{(1)^2 + (1)^2 + (-1)^2}$ $= \sqrt{3}$	$\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{bmatrix}$

$$N = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

To find Diagonal Matrix D:

$$D = N^T A N$$

$$= \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{12}{\sqrt{6}}+\frac{2}{\sqrt{6}}+\frac{2}{\sqrt{6}} & 0-\frac{2}{\sqrt{2}}+\frac{2}{\sqrt{2}} & \frac{6}{\sqrt{3}}-\frac{2}{\sqrt{3}}-\frac{2}{\sqrt{3}} \\ \frac{-4}{\sqrt{6}}-\frac{3}{\sqrt{6}}-\frac{1}{\sqrt{6}} & 0+\frac{3}{\sqrt{2}}-\frac{1}{\sqrt{2}} & \frac{-2}{\sqrt{3}}+\frac{3}{\sqrt{3}}+\frac{1}{\sqrt{3}} \\ \frac{4}{\sqrt{6}}+\frac{1}{\sqrt{6}}+\frac{3}{\sqrt{6}} & 0-\frac{1}{\sqrt{2}}+\frac{3}{\sqrt{2}} & \frac{2}{\sqrt{3}}-\frac{1}{\sqrt{3}}-\frac{3}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{16}{\sqrt{6}} & 0 & \frac{2}{\sqrt{3}} \\ \frac{-8}{\sqrt{6}} & \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{3}} \\ \frac{8}{\sqrt{6}} & \frac{2}{\sqrt{2}} & \frac{-2}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{32}{\sqrt{36}}+\frac{8}{\sqrt{36}}+\frac{8}{\sqrt{36}} & 0-\frac{2}{\sqrt{12}}+\frac{2}{\sqrt{12}} & \frac{4}{\sqrt{18}}-\frac{2}{\sqrt{18}}-\frac{2}{\sqrt{18}} \\ 0-\frac{8}{\sqrt{12}}+\frac{8}{\sqrt{12}} & 0+\frac{2}{\sqrt{4}}+\frac{2}{\sqrt{4}} & 0+\frac{2}{\sqrt{6}}-\frac{2}{\sqrt{6}} \\ \frac{16}{\sqrt{18}}-\frac{8}{\sqrt{18}}-\frac{8}{\sqrt{18}} & 0+\frac{2}{\sqrt{6}}-\frac{2}{\sqrt{6}} & \frac{2}{\sqrt{9}}+\frac{2}{\sqrt{9}}+\frac{2}{\sqrt{9}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{48}{\sqrt{36}} & 0 & 0 \\ 0 & \frac{4}{\sqrt{4}} & 0 \\ 0 & 0 & \frac{6}{\sqrt{9}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{48}{6} & 0 & 0 \\ 0 & \frac{4}{2} & 0 \\ 0 & 0 & \frac{6}{3} \end{bmatrix}$$

$$D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

To find canonical form:

$$Y^T DY = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 8y_1^2 + 2y_2^2 + 2y_3^2$$

To find Rank, index, signature and nature:

Rank = Number of square terms.

=3

Index = Number of positive square terms.

= 3

Signature = Number of positive square terms – Number of negative square terms.

= 3−0

= 3

Nature = positive definite.

22. Reduce the Quadratic form $2x_1x_2 + 2x_1x_3 - 2x_2x_3$ to a canonical form by an orthogonal reduction and discuss its nature. Also find the model matrix. [G]

(Or)

Reduce the Quadratic form $2x_1x_2 + 2x_1x_3 - 2x_2x_3$ to canonical form through an orthogonal transformation and find the rank, index, signature and nature of the quadratic form

[A/M 2011 R2008][M/J 2014 R2008]

<u>Answer:</u>

<u>To find Matrix:</u>

Given Equation, $2x_1x_2 + 2x_1x_3 - 2x_2x_3$

$$Matrix = \begin{bmatrix} coeff\ x_1^2 & \frac{1}{2}coeff\ x_1x_2 & \frac{1}{2}coeff\ x_1x_3 \\ \frac{1}{2}coeff\ x_1x_2 & coeff\ x_2^2 & \frac{1}{2}coeff\ x_2x_3 \\ \frac{1}{2}coeff\ x_1x_3 & \frac{1}{2}coeff\ x_2x_3 & coeff\ x_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{2}(2) & \frac{1}{2}(2) \\ \frac{1}{2}(2) & 0 & \frac{1}{2}(-2) \\ \frac{1}{2}(2) & \frac{1}{2}(-2) & 0 \end{bmatrix}$$

Matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

<u>To find Characteristic equation:</u>

$S_1 = Sum\ of\ diagonal\ elements$

$= 0 + 0 + 0$

$\boxed{S_1 = 0}$

$S_2 = Sum\ of\ minor\ of\ diagonal\ elements$

$$= \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$= [0 - 1] + [0 - 1] + [0 - 1]$

$\boxed{S_2 = -3}$

$S_3 = |A|$

$$= \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= 0\begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix}$$

$$= 0 - [0+1] + [-1-0]$$

$$= -1 - 1$$

$$\boxed{S_3 = -2}$$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$$\lambda^3 - 0\lambda^2 + (-3)\lambda - (-2) = 0$$

$$\lambda^3 - 0\lambda^2 - 3\lambda + 2 = 0$$

To find Eigen values:

$$\begin{array}{r|rrrr} \lambda = 1 & 1 & 0 & -3 & 2 \\ & 0 & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & \underline{|0} \\ & \lambda^2 & \lambda & const & \end{array}$$

$$\lambda^2 + \lambda - 2 = 0$$

$$(\lambda - 1)(\lambda + 2) = 0 \quad \begin{bmatrix} \because sum = 1 \text{ and } product = -2 \\ (-1,2) \end{bmatrix}$$

$$\lambda = 1, \lambda = -2$$

Eigen values are −2, 1, 1

To find Eigen vector:

Case (i) λ = −2:

$$[A - \lambda I]X = 0$$

$$[A - (-2)I]X = 0 \quad \{\because \lambda = -2\}$$

$$[A + 2I]X = 0$$

$$\begin{bmatrix} 0+2 & 1 & 1 \\ 1 & 0+2 & -1 \\ 1 & -1 & 0+2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Select last two rows

$$\begin{matrix} & x_1 & x_2 & x_3 & \\ 2 & -1 & 1 & 2 \\ -1 & 2 & 1 & -1 \end{matrix}$$

$$\frac{x_1}{4-1} = \frac{x_2}{-1-2} = \frac{x_3}{-1-2}$$

$$\frac{x_1}{3} = \frac{x_2}{-3} = \frac{x_3}{-3}$$

Divided by 3

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

Eigen vector $X_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

Case (ii) λ = 1:

$$[A - \lambda I]X = 0$$

$$[A - 1.I]X = 0 \quad \{\because \lambda = 1\}$$

$$\begin{bmatrix} 0-1 & 1 & 1 \\ 1 & 0-1 & -1 \\ 1 & -1 & 0-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \{R_1 \leftrightarrow -R_1\}$$

$$x_1 - x_2 - x_3 = 0 \rightarrow (*)$$

Put $x_1 = 0$ in $(*)$	Put $x_2 = 0$ in $(*)$
$-x_2 - x_3 = 0$	$x_1 - x_3 = 0$
$-x_2 = x_3$	$x_1 = x_3$
$\frac{x_2}{1} = \frac{x_3}{-1}$	$\frac{x_1}{1} = \frac{x_3}{1}$
Eigen vector $X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$	**Eigen vector** $X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

To check orthogonal property:

$$X_1^T X_2 = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 - 1 + 1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} = 0$$

$$X_2^T X_3 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 + 0 - 1 \end{bmatrix} = \begin{bmatrix} -1 \end{bmatrix} \neq 0$$

$$X_3^T X_1 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 + 0 - 1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} = 0$$

X_1, X_2, X_3 are not orthogonal in pairs.

Choose $X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ **such that** $X_2^T X_3 = 0$ **and** $X_3^T X_1 = 0$

$$\begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \text{ and} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = 0$$

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 1 & & -1 & & 0 & & 1 \\ -1 & & -1 & & 1 & & -1 \end{array}$$

$$\frac{x_1}{-1-1} = \frac{x_2}{-1-0} = \frac{x_3}{0-1}$$

$$\frac{x_1}{-2} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

Divided by (−)

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{1}$$

Eigen vector $X_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

<u>To find Model Matrix N:</u>

Eigen vector	$\sqrt{x_1^2 + x_2^2 + x_3^2}$	Normalized Eigen vector

$X_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$	$\sqrt{1^2 + (-1)^2 + (-1)^2}$ $= \sqrt{3}$	$\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{bmatrix}$
$X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$	$\sqrt{(0)^2 + (1)^2 + (-1)^2}$ $= \sqrt{2}$	$\begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$
$X_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$	$\sqrt{(2)^2 + (1)^2 + (1)^2}$ $= \sqrt{6}$	$\begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$

$$N = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

To find Diagonal Matrix D:

$$D = N^T AN$$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 0 - \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} & 0 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} & 0 + \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} + 0 + \frac{1}{\sqrt{3}} & 0 + 0 + \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + 0 & 0 - \frac{1}{\sqrt{2}} + 0 & \frac{2}{\sqrt{6}} - \frac{1}{\sqrt{6}} + 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{-2}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-2}{\sqrt{9}} - \frac{2}{\sqrt{9}} - \frac{2}{\sqrt{9}} & 0 - \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{18}} - \frac{1}{\sqrt{18}} - \frac{1}{\sqrt{18}} \\ 0 + \frac{2}{\sqrt{6}} - \frac{2}{\sqrt{6}} & 0 + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{4}} & 0 + \frac{1}{\sqrt{12}} - \frac{1}{\sqrt{12}} \\ \frac{-4}{\sqrt{18}} + \frac{2}{\sqrt{18}} + \frac{2}{\sqrt{18}} & 0 + \frac{1}{\sqrt{12}} - \frac{1}{\sqrt{12}} & \frac{4}{\sqrt{36}} + \frac{1}{\sqrt{36}} + \frac{1}{\sqrt{36}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-6}{\sqrt{9}} & 0 & 0 \\ 0 & \frac{2}{\sqrt{4}} & 0 \\ 0 & 0 & \frac{6}{\sqrt{36}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-6}{3} & 0 & 0 \\ 0 & \frac{2}{2} & 0 \\ 0 & 0 & \frac{6}{6} \end{bmatrix}$$

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To find canonical form:

$$Y^T DY = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= -2y_1^2 + 1y_2^2 + 1y_3^2$$

To find Rank, index, signature and nature:

Rank = Number of square terms.

=3

Index = Number of positive square terms.

= 2

Signature = Number of positive square terms – Number of negative square terms.

= 2−1

= 1

Nature = Indefinite.