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UNIT I – MATRICES
PART – A

1. **Find the sum and product of all the eigen values of** $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ **(APR/MAY 2015)**

Solution : Sum of the eigen values = Sum of the main diagonal elements = 8+7+3 = 18

Product of the eigen values = $|A|$ = 8(5)+6(-10)+2(10) = 0

2. **If 1 and 2 are the two eigen values of** $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$**, find** $|A|$ **without expanding the determinant.**

Solution : Let λ be the third eigen value of the given matrix.

We know that, sum of the eigen values = sum of the main diagonal elements.

i.e. $1 + 2 + \lambda = 2+2+2 \Rightarrow \lambda = 3$

Now, $|A|$ = product of all eigen values = (1)(2)(3) = 6

3. **The product of two eigen values of the matrix** $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ **is 16. Find the third eigen value.**

(DEC/JAN 2011, APR/MAY 2012)

Solution : Let λ_1 , λ_2 , λ_3 be the eigen values of the given matrix.

We know that, product of the eigen values = $|A|$

i.e. $\lambda_1 \lambda_2 \lambda_3 = |A|$

$(16)\ \lambda_3 = 6(9–1) + 2(–6 +2) + 2(2–6)$ [since the product of two eigen values is 16]

$(16)\ \lambda_3 = 32 \Rightarrow \lambda_3 = 2$

4. **One of the eigen values of** $\begin{bmatrix} 7 & 4 & 4 \\ 4 & -8 & -1 \\ 4 & -1 & -8 \end{bmatrix}$ **is –9, find the other two eigen values.**

Solution : Let λ_1, λ_2 be the other two eigen values.

We know that, sum of the eigen values = sum of the main diagonal elements

i.e. $\lambda_1 + \lambda_2 - 9 = 7 - 8 - 8 = -9$

$\lambda_1 + \lambda_2 = 0 \Rightarrow \lambda_1 = -\lambda_2$ …(1)

We know that, product of the eigen values = $|A|$

$-9\lambda_1 \lambda_2 = |A| = 441$

$\lambda_1 \lambda_2 = -49 \Rightarrow \lambda_1 = \frac{-49}{\lambda_2}$ … (2)

substitute in (1) we get, $-\lambda_2 = \frac{-49}{\lambda_2}$

$\lambda_2^{\ 2} = 49 \Rightarrow \lambda_2 = \pm 7$

(1) $\Rightarrow \lambda_1 = \mp 7$. Hence the other two eigen values are 7 and -7.

5. Find the eigen values of the inverse of the matrix $A=\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix}$ (APR/MAY 2014)

Solution : In a triangular matrix, the main diagonal values are the eigen values of the matrix.

$\therefore$ 2, 3, 4 are the eigen values of A. Hence the eigen values of A^{-1} = $\frac{1}{2},\frac{1}{3},\frac{1}{4}$.

6. If the eigen values of the matrix A of order 3 × 3 matrix are 2,3 and 1, then find the eigen values of adjoint of A. (NOV/DEC 2014)

Solution : We know that, adjoint of A = $A^{-1}|A|$.

$|A|$ = product of the eigen values = (2)(3)(1) = 6.

Eigen values of $A^{-1} = \frac{1}{2},\frac{1}{3},1$.

$\therefore$ Eigen values of $\text{adj}\,A = \frac{1}{2}(6),\frac{1}{3}(6),(1)(6) = 3,2,6$

7. State Cayley Hamilton theorem. (NOV/DEC 2014)

Statement : Every square matrix satisfies its own characteristic equation.

8. Given $A=\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, Find A^{-1} using Cayley – Hamilton theorem.

Solution : The characteristic equation is $\lambda^2 - S_1\lambda + S_2 = 0$,

Here, $S_1 = 4$ and $S_2 = -5 \Rightarrow \lambda^2 - 4\lambda - 5 = 0$.

By Cayley – Hamilton theorem $A^2 - 4A - 5I = 0$.

Multiply by A^{-1}, we get $A - 4I - 5A^{-1} = 0$ $\therefore$ $A^{-1} = \frac{1}{5}[A-4I] = \begin{bmatrix} \frac{-3}{5} & \frac{2}{5} \\ \frac{4}{5} & \frac{-1}{5} \end{bmatrix}$

9. If $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ is an eigen vector of $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$, find the corresponding eigen value.

Solution : $(A - \lambda I)X = 0 \Rightarrow \begin{pmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow (-2-\lambda)(1)+2(2)+(-3)(-1) = 0 \Rightarrow \lambda = 5.$

10. Find the eigen vector of $A=\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ corresponding to the eigen value 2.

Solution : Let X = $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the eigen vector of the matrix corresponding to the eigen value λ.

The eigen vectors are obtained from the equation $(A - \lambda I)X = 0$

$$\Rightarrow \begin{pmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

When $\lambda = 2$, $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x_2 = 0, x_3 = 0$ and x_1 takes any value, say $k \neq 0$.

Therefore the eigenvector is $\begin{pmatrix} k \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

11. Find the constants 'a' & 'c' such that the matrix $\begin{pmatrix} a & 4 \\ 1 & c \end{pmatrix}$ has 3 & –2 as its eigen values.

Solution : **(APR/MAY 2011, 2017)**

Sum of the eigen values = sum of the main diagonals $\Rightarrow a + c = 3 - 2 = 1$-----(1)

product of the eigen values = $|A| \Rightarrow (3)(-2) = ac - 4$

i.e. $-6 = ac - 4 \Rightarrow ac = -2$

$\therefore c = -2/a$

sub c in (1) $a + c = 1 \Rightarrow a + (-2/a) = 1 \Rightarrow a^2 - 2 = a$ i.e. $a^2 - a - 2 = 0$

solving $a = -1, 2 \Rightarrow c = 2, -1$

12. Determine λ so that $\lambda(x^2 + y^2 + z^2) + 2xy - 2xz + 2zy$ is positive definite.

Solution : The matrix of the given quadratic form is $A = \begin{pmatrix} \lambda & 1 & -1 \\ 1 & \lambda & 1 \\ -1 & 1 & \lambda \end{pmatrix}$

The principal sub determinants are given by

$D_1 = \lambda$, $D_2 = \begin{vmatrix} \lambda & 1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda + 1)(\lambda - 1)$ & $D_3 = |A| = (\lambda+1)^2(\lambda - 2)$

The Quadratic form is +ve definite if D_1, D_2 & $D_3 > 0 \Rightarrow \lambda > 2$

13. If λ is the eigen value of the matrix A, then prove that λ^2 is the eigen value of A^2.

Solution : **(JAN 2014)**

Let X be the eigen vector of the matrix A corresponding to the eigen value λ, then $AX = \lambda X$.

Multiply by $A \Rightarrow A^2 X = A(\lambda X)$

$= \lambda(AX)$

$= \lambda(\lambda X)$

$= \lambda^2 X$

Hence, λ^2 is the eigen value of A^2.

14. What is the nature of the quadratic form $x^2 + y^2 + z^2$ in four variables? (JAN 2016)

Solution : The matrix of the given quadratic form is $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Since the matrix is the diagonal matrix, its main diagonal elements are its eigen values.

$\therefore$ The eigen values are 1,1,1,0. Hence the nature is positive semi definite.

15. If 2,-1,-3 are the eigen values of the matrix A, find the eigen values of the matrix $A^2 - 2I$.

Solution : **(A/M 2014)**

The eigen values of A^2 are $2^2, (-1)^2, (-3)^2 = 4, 1, 9$.

The eigen values of $A^2 - 2I$ are $4 - 2, 1 - 2, 9 - 2 = 2, -1, 7$

16. If 2,3 are the two eigen values of $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ b & 0 & 2 \end{pmatrix}$,then find the value of b. **(NOV/DEC 2014)**

Solution: Let λ be the third eigen value of the given matrix.
Sum of the eigen values = sum of the main diagonals
i.e. $2+3+\lambda = 6 \Rightarrow \lambda = 1$.
product of the eigen values = $|A|$
$(1)(2)(3) = 2(4) +1(-2b) \Rightarrow 6 = 8-2b \Rightarrow b=1$

17. Find the rank, index and signature of the Quadratic form whose Canonical form is $x_1^2 + 2x_2^2 - 3x_3^2$. **(APR/MAY 2011)**

Solution :
Rank (r) = Number of terms in the C.F = 3 ,
Index (p) = Number of Positive terms in the C.F = 2
Signature (s) = 2p – r =1

18. Identify the nature, index and signature of the quadratic form $2x_1x_2 + 2x_2x_3 + 2x_3x_1$. **(NOV/DEC 2015)**

Solution:

The matrix of the quadratic form is given by $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

The characteristics equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.
S_1 = Sum of the main diagonal elements = 0
S_2 = Sum of the minors of the main diagonal element $=(0-1)+(0-1)+(0-1)=-3$;
$S_3 = |A| = -1(0-1)+1(1-0)=2$
The characteristics equation is $\lambda^3 - 3\lambda - 2 = 0$.
$(\lambda+1)^2(\lambda-2)=0 \Rightarrow$ The eigen values are $\lambda = -1,-1,2$
Nature: indefinite
Rank (r) = Number of eigen values.
Index (p) = Number of Positive eigen values.
Signature (s) = 2p – r =2(1) – 3 = –1.

19. Write down the matrix of the quadratic form $2x^2 + 8z^2 + 4xy + 10xz - 2yz$. **(APR/MAY 2013)**

Solution :
The matrix of the quadratic form is given by
a_{11}= coeff of x^2= 2 , a_{22}= coeff of y^2 = 0 , a_{33} = coeff of z^2= 8

$a_{12}=a_{21}= \frac{1}{2}$(coeff of xy) = $\frac{4}{2}$ =2, $a_{13}= a_{31}= \frac{1}{2}$(coeff of xz) = $\frac{10}{2}$ = 5

$a_{23}=a_{32} = \frac{1}{2}$(coeff of yz) = $\frac{-2}{2}$ =-1

$$\Rightarrow A = \begin{bmatrix} 2 & 2 & 5 \\ 2 & 0 & -1 \\ 5 & -1 & 8 \end{bmatrix}$$

20. Write down the quadratic form corresponding to the matrix $A = \begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}$. **(APR/MAY 2012)**

Solution :

Quadratic form of A is given by $X^TAX = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$= 0x_1^2 + x_2^2 + 2x_3^2 + 10x_1x_2 + 12x_2x_3 - 2x_3x_1$$

PART-B

1(a) Find the eigen values and eigen vectors of the matrix $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ **(JAN 2016)**

Solution:

The characteristics equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

S_1 = Sum of the main diagonal elements = 2+2+2 = 6;

S_2 = Sum of the minors of the main diagonal element

$$= \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = (4-0)+(4-1)+(4-0) = 11;$$

$$S_3 = |A| = 2\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = 2(4-0)+1(0-2) = 8-2 = 6.$$

The characteristics equation is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$.

$(\lambda-1)(\lambda-2)(\lambda-3) = 0 \Rightarrow$

The eigen values are $\lambda = 1,2,3$

We know that, $(A-\lambda I)X=0 \Rightarrow \begin{pmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$$\left.\begin{aligned} (2-\lambda)x_1 + 0x_2 + x_3 &= 0 \\ 0x_1 + (2-\lambda)x_2 + 0x_3 &= 0 \\ +x_1 + 0x_2 + (2-\lambda)x_3 &= 0 \end{aligned}\right\} ----(1)$$

Case (1) : $\lambda = 1$

Substituting $\lambda=1$ in (1) we get

$x_1 + x_3 = 0,$

$x_2 = 0,$

$x_1 + x_3 = 0$

Solving $x_1 = -x_3, x_2 = 0.$

Take $x_1 = 1 \Rightarrow x_3 = -1.$

$\therefore$ The eigen vectors is $X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Case (2) : $\lambda = 2$

Substituting $\lambda=2$ in (1) we get

$x_3 = 0,$

$x_1 = 0$

$x_1 = x_3 = 0$ and x_2 takes any value . Take $x_2 = 1$

The eigen vector is $X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Case (3) : $\lambda = 3$

Substituting $\lambda=3$ in (1) we get

$-x_1 + x_3 = 0,$

$x_2 = 0,$

$x_1 - x_3 = 0$

Solving $x_1 = x_3, x_2 = 0.$

Take $x_1 = 1 \Rightarrow x_3 = 1.$

$\therefore$ The eigen vector is $X_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

The eigen vectors are $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

(b) Find the eigen values and eigen vectors of the matrix $A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$ **(APR/MAY 2014)**

Solution:

The characteristic equation is $\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0.$

S_1 = Sum of the main diagonal elements = –1;

S_2 = Sum of the minors of the main diagonal elements = -21;

$S_3 = |A| = 45.$

The characteristic equation is $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0.$

$(\lambda + 3)(\lambda + 3)(\lambda - 5) = 0$

The eigen values are $\lambda = -3, -3, 5$

Consider $\begin{pmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$$\left.\begin{aligned} (-2-\lambda)x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 + (1-\lambda)x_2 - 6x_3 &= 0 \\ -x_1 - 2x_2 - \lambda x_3 &= 0 \end{aligned}\right\} ----(1)$$

Case (1) : $\boldsymbol{\lambda = 5}$

Substituting $\lambda=5$ in (1) we get

$-7x_1 + 2x_2 - 3x_3 = 0,$

$2x_1 - 4x_2 - 6x_3 = 0,$

$-x_1 - 2x_2 - 5x_3 = 0$

Solving the above equation we get $x_1 = -1, x_2 = -2, x_3 = 1$

Eigen vectors is $X_1 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$

Case (2) : **$\lambda = -3$**

Substituting $\lambda = -3$ in (1) we get

$x_1 + 2x_2 - 3x_3 = 0,$

$2x_1 + 4x_2 - 6x_3 = 0,$

$x_1 + 2x_2 - 3x_3 = 0,$

The above three equations are reduced to single equation $x_1 + 2x_2 - 3x_3 = 0,$

Put $x_2 = 0$ we get the value , $x_1 = 3 \quad x_3 = 1 \quad \therefore X_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

Put $x_3 = 0$ we get the value , $x_1 = -2 \quad x_2 = 1 \quad \therefore X_3 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

Therefore the Eigen values $\lambda = 5, -3, -3$ with corresponding Eigen vector are

$\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

2(a) Diagonalize the matrix A= $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ **by means of an orthogonal transformation.**

Solution:

The symmetric matrix A= $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$

The characteristic equation is $\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0.$

S_1 = Sum of the main diagonal elements = 18;

S_2 = Sum of the minors of the main diagonal elements = 45;

$S_3 = |A| = 0.$

The characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda = 0.$

$(\lambda)(\lambda-3)(\lambda-15) = 0 \Rightarrow$ The eigen values are $\lambda = 0, 3, 15$

Consider $\begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$ -----(1)

Case (1) : **$\lambda = 0$**

Substituting $\lambda=0$ in (1) we get

$8x_1 - 6x_2 + 2x_3 = 0,$

$6x_1 + 7x_2 - 4x_3 = 0,$

$2x_1 - 4x_2 + 3x_3 = 0$

Solving $x_1 = 1,\ x_3 = 2, x_2 = 2,$ $\Rightarrow$ Eigen vectors is $X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

Case (2) : **$\lambda = 3$**

Substituting $\lambda=3$ in the (1) and solving we get

$5x_1 - 6x_2 + 2x_3 = 0, -6x_1 + 4x_2 - 4x_3 = 0,\ 2x_1 - 4x_2 = 0$

solving $x_1 = 2,\ x_3 = -2, x_2 = 1,$ $\Rightarrow$ Eigen vector is $X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

Case (3) : **$\lambda = 15$**

Substituting $\lambda=15$ in the (1) and solving we get

$-7x_1 - 6x_2 + 2x_3 = 0,$

$-6x_1 - 8x_2 - 4x_3 = 0,$

$2x_1 - 4x_2 - 12x_3 = 0$

Solving the above equation (by cross ratio) we get $x_1 = 2,\ x_3 = 1, x_2 = -2.$

Eigen vector is $X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

It is clear that $X_1^T X_2 = X_1^T X_3 = X_2^T X_3 = 0$

Normalized Eigen vectors are $\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{-2}{3} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{-2}{3} \\ \frac{1}{3} \end{pmatrix}$

Normalized matrix $N = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$ and $N^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$

$$N^TAN=\begin{bmatrix}\frac{1}{3}&\frac{2}{3}&\frac{2}{3}\\\frac{2}{3}&\frac{1}{3}&-\frac{2}{3}\\\frac{2}{3}&-\frac{2}{3}&\frac{1}{3}\end{bmatrix}\begin{bmatrix}8&-6&2\\-6&7&-4\\2&-4&3\end{bmatrix}\begin{bmatrix}\frac{1}{3}&\frac{2}{3}&\frac{2}{3}\\\frac{2}{3}&\frac{1}{3}&-\frac{2}{3}\\\frac{2}{3}&-\frac{2}{3}&\frac{1}{3}\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&3&0\\0&0&15\end{bmatrix}=D\,(0,3,15)$$

(b) **The Eigen vectors of a 3×3 real symmetric matrix A corresponding to the eigen values 2,3,6 are $(1,0,-1)^T$, $(1,1,1)^T$ and $(1,-2,1)^T$ respectively. Find the matrix A.** **(APR/MAY 2011)**

Solution:

We know that under orthogonal transformation real symmetric matrix A can be diagonalized in to a diagonal matrix $D=\begin{bmatrix}2&0&0\\0&3&0\\0&0&6\end{bmatrix}$.

i.e. $N^TAN = D$, where N is an orthogonal matrix.

Pre multiply by N and post multiply by N^T, we get

$$N(N^TAN)N^T = N(D)N^T$$

$$(NN^T)A(NN^T) = NDN^T$$

$$I(A)I = NDN^T$$

$$\therefore A = NDN^T$$

Normalized matrix $N=\begin{bmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt3}&\frac{1}{\sqrt6}\\\frac{0}{\sqrt2}&\frac{1}{\sqrt3}&-\frac{2}{\sqrt6}\\\frac{-1}{\sqrt2}&\frac{1}{\sqrt3}&\frac{1}{\sqrt6}\end{bmatrix}$ and $N^T=\begin{bmatrix}\frac{1}{\sqrt2}&\frac{0}{\sqrt2}&\frac{-1}{\sqrt2}\\\frac{1}{\sqrt3}&\frac{1}{\sqrt3}&\frac{1}{\sqrt3}\\\frac{1}{\sqrt6}&\frac{-2}{\sqrt6}&\frac{1}{\sqrt6}\end{bmatrix}$

$$A=NDN^T=\begin{bmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt3}&\frac{1}{\sqrt6}\\\frac{0}{\sqrt2}&\frac{1}{\sqrt3}&-\frac{2}{\sqrt6}\\\frac{-1}{\sqrt2}&\frac{1}{\sqrt3}&\frac{1}{\sqrt6}\end{bmatrix}\begin{bmatrix}2&0&0\\0&3&0\\0&0&6\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt2}&\frac{0}{\sqrt2}&\frac{-1}{\sqrt2}\\\frac{1}{\sqrt3}&\frac{1}{\sqrt3}&\frac{1}{\sqrt3}\\\frac{1}{\sqrt6}&\frac{-2}{\sqrt6}&\frac{1}{\sqrt6}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{2}{\sqrt2}&\frac{3}{\sqrt3}&\frac{6}{\sqrt6}\\0&\frac{3}{\sqrt3}&-\frac{2}{\sqrt6}\\\frac{-2}{\sqrt2}&\frac{3}{\sqrt3}&\frac{6}{\sqrt6}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt2}&\frac{0}{\sqrt2}&\frac{-1}{\sqrt2}\\\frac{1}{\sqrt3}&\frac{1}{\sqrt3}&\frac{1}{\sqrt3}\\\frac{1}{\sqrt6}&\frac{-2}{\sqrt6}&\frac{1}{\sqrt6}\end{bmatrix}=\begin{bmatrix}3&-1&1\\-1&5&-1\\1&-1&3\end{bmatrix}$$

$$\therefore A=\begin{bmatrix}3&-1&1\\-1&5&-1\\1&-1&3\end{bmatrix}$$

3 **Reduce the quadratic form $x_1^2+5x_2^2+x_3^2+2x_1x_2+2x_2x_3+6x_3x_1$ into a canonical form by using orthogonal transformation. Hence find its rank, index, signature and nature. (APR/MAY 2014, 2015)**

Solution:

The matrix of the quadratic form is given by $A=\begin{bmatrix}1&1&3\\1&5&1\\3&1&1\end{bmatrix}$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

S_1 = Sum of the main diagonal elements =1+5+1 = 7;

S_2 = Sum of the minors of the main diagonal elements = (5-1)+(1-9)+(5-1) = 4–8+4 = 0;

$S_3 = |A| = -36$.

The characteristic equation is $\lambda^3 - 7\lambda^2 + 36 = 0$.

$(\lambda+2)(\lambda-3)(\lambda-6) = 0$

$\therefore$ The eigen values are $\lambda = -2, 3, 6$

Consider $\begin{bmatrix}1-\lambda&1&3\\1&5-\lambda&1\\3&1&1-\lambda\end{bmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0$

$$\left.\begin{aligned}(1-\lambda)x_1+x_2+3x_3=0\\x_1+(5-\lambda)x_2+x_3=0\\3x_1+x_2+(1-\lambda)x_3=0\end{aligned}\right\}----(1)$$

Case (1) : **$\lambda = -2$**

Substituting $\lambda = -2$ in (1) we get

$3x_1+x_2+3x_3=0$

$x_1+7x_2+x_3=0$

$3x_1+x_2+3x_3=0$

Solving the above equation (by cross ratio) we get $x_1=-1, x_2=0, x_3=1$

Eigen vectors is $X_1=\begin{pmatrix}-1\\0\\1\end{pmatrix}$

Case (2) : **$\lambda = 3$**

Substituting $\lambda = 3$ in (1) we get

$-2x_1+x_2+3x_3=0$

$x_1+2x_2+x_3=0$

$3x_1+x_2-2x_3=0$

Solving the above equation (by cross ratio) we get $x_1=1, x_2=-1, x_3=1$

Eigen vectors is $X_2=\begin{pmatrix}1\\-1\\1\end{pmatrix}$

Case (3) : **$\lambda = 6$**

Substituting $\lambda = 6$ in (1) we get

$-5x_1 + x_2 + 3x_3 = 0$

$x_1 - x_2 + x_3 = 0$

$3x_1 + x_2 - 5x_3 = 0$

Solving the above equation (by cross ratio) we get $x_1 = 1, x_2 = 2, x_3 = 1$

Eigen vectors is $X_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$\therefore$ The eigen vectors are $X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, X_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, X_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

It is clear that $X_1^T X_2 = X_1^T X_3 = X_2^T X_3 = 0$. $\Rightarrow$ all the eigen vectors are pairwise orthogonal.

Normalized modal matrix $N = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{0}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$ and $N^T = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

$$N^T AN = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{0}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} = D(-2, 3, 6)$$

Consider the orthogonal transformation X= NY, where N is an orthogonal matrix.

Now, Quadratic form = $X^T AX = (NY)^T A(NY)$

$= (Y^T N^T)A(NY)$

$= Y^T (N^T AN)Y$

$= Y^T (D)Y$ = Canonical form

Under orthogonal transformation X = NY the given quadratic form reduced to canonical form provided $N^T AN = D$.

$\therefore$ Reduced canonical form is $-2y_1^2 + 3y_2^2 + 6y_3^2$.

Nature: indefinite

Rank (r) = Number of terms in the C.F = 3.

Index (p) = Number of Positive terms in the C.F = 2.

Signature (s) = 2p – r =2(2) – 3 = 1.

4(a) Reduce the quadratic form $6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4xz$ **into a canonical form by using orthogonal transformation. Hence find its rank, index, signature and nature.** **(NOV/DEC 2015, JAN 2014)**

Solution:

The matrix of the quadratic form is given by $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

The characteristic equation is $\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0$.

S_1 = Sum of the main diagonal elements =6+3+3 = 12;

S_2 = Sum of the minors of the main diagonal elements = (9-1)+(18-4)+(18-4) = 8+14+14 = 36;

$S_3 = |A|$ = 32.

The characteristic equation is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$.

$(\lambda - 2)(\lambda - 2)(\lambda - 8) = 0$

$\therefore$ The eigen values are λ = 8,2,2

Consider $\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$$\left.\begin{aligned} (6-\lambda)x_1 - 2x_2 + 2x_3 &= 0 \\ -2x_1 + (3-\lambda)x_2 - x_3 &= 0 \\ 2x_1 - x_2 + (3-\lambda)x_3 &= 0 \end{aligned}\right\} ----(1)$$

Case (1) : **λ = 8**

Substituting λ= 8 in (1) we get

$-2x_1 - 2x_2 + 2x_3 = 0$

$-2x_1 - 5x_2 - x_3 = 0$

$2x_1 - x_2 - 5x_3 = 0$

Solving the above equation (by cross ratio) we get $x_1 = 2, x_2 = -1, x_3 = 1$

Eigen vectors is $X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

Case (2) : **λ = 2**

Substituting λ= 2 in (1) we get

$4x_1 - 2x_2 + 2x_3 = 0$

$-2x_1 + x_2 - x_3 = 0$

$2x_1 - x_2 + x_3 = 0$

All the above equations are reduced to single equation $2x_1 - x_2 + x_3 = 0$.

Assume $x_1 = 0, x_2 = 1$. Substitute in $2x_1 - x_2 + x_3 = 0$ we get, $x_3 = 1$.

Eigen vectors is $X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Case (3) : **λ = 2**

In order to get the pairwise orthogonal eigen vectors we assume the third eigen vector as

$X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Since $X_3^T X_1 = 0$ we get $2a - b + c = 0$

Since $X_2^T X_3 = 0$ we get $0a + b + c = 0$

Solving the above equation (by cross ratio) we get $a = 1, b = 1, c = -1$

Eigen vectors is $X_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$\therefore$ The eigen vectors are $X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, X_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

Normalized modal matrix N= $\begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$ and $N^T = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$

$$N^TAN = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D$$

Consider the orthogonal transformation X= NY, where N is a orthogonal matrix.

Now, Quadratic form = $X^TAX = (NY)^TA(NY)$

$= (Y^T N^T)A(NY)$

$= Y^T (N^TAN)Y$

$= Y^T (D)Y$ = Canonical form

$\therefore$ Under orthogonal transformation X = NY the given quadratic form reduced to canonical form provided $N^TAN = D$. Reduced canonical form is $8y_1^2 + 2y_2^2 + 2y_3^2$.

Nature: Positive definite

Rank (r) = Number of terms in the C.F = 3.

Index (p) = Number of Positive terms in the C.F = 3.

Signature (s) = 2p – r =2(3) – 3 = 3.

5(a) Verify Cayley- Hamilton theorem for the matrix A= $\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$ **and hence find A^4 and A^{-1}.**

(NOV/DEC 2014, APR/MAY 2017)

Solution:

The characteristics equation is $\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0$.

S_1 = Sum of the main diagonal elements = 6;

S_2 = Sum of the minors of the main diagonal elements = 8;

$S_3 = |A| = 3$.

The characteristic equation is $\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$.

To verify Cayley Hamilton theorem, we have to show that $A^3 - 6A^2 + 8A - 3I = 0$

$$A^2 = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix}\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{pmatrix}$$

$$A^3 - 6A^2 + 8A - 3I = \begin{pmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{pmatrix} - 6\begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix} + 8\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ Cayley Hamilton Theorem is verified.

Find A^4

Pre multiply (1) by A

$$(A^3 - 6A^2 + 8A - 3I)A = 0 \qquad \Rightarrow A^4 - 6A^3 + 8A^2 - 3A = 0$$

$$\Rightarrow A^4 = 6A^3 - 8A^2 + 3A$$

$$A^4 = 6\begin{pmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{pmatrix} - 8\begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix} + 3\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 124 & 123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{pmatrix}$$

Find A^{-1}

Pre multiply (1) by A^{-1}

$$A^{-1}(A^3 - 6A^2 + 8A - 3I) = 0 \qquad \Rightarrow A^2 - 6A + 8I - 3A^{-1} = 0$$

$$\Rightarrow A^{-1} = A^2 - 6A + 8I$$

$$A^{-1} = \begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix} - 6\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} + 8\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{pmatrix}$$

(b) Using Cayley-Hamilton theorem, find the matrix represented by

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 - 8A^2 + 2A - I$ **when** $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$. **(NOV/DEC 2015)**

Solution:

The characteristics equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

S_1 = Sum of the main diagonal elements = 5;

S_2 = Sum of the minors of the main diagonal elements = 7;

$S_3 = |A| = 3.$

Characteristic equation : $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

Cayley-Hamilton theorem states that, every Square matrix satisfies its own characteristic equation.

$\Rightarrow A^3 - 5A^2 + 7A - 3I = 0.$

Let P(A) = $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 - 8A^2 + 2A - I$

D(A) = $A^3 - 5A^2 + 7A - 3I$

By long division method we get ,

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 - 8A^2 + 2A - I$$

$$= \left(A^3 - 5A^2 + 7A - 3I\right)(A^5 + A) + A^2 + A + I$$

$$= (0)\ (A^5 + A) + A^2 - A + I = A^2 + A + I$$

$$A^2 + A + I = \begin{pmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{pmatrix}$$

$$\therefore A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 - 8A^2 + 2A - I = \begin{pmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{pmatrix}$$

UNIT II - VECTOR CALCULUS
PART A

1. Find $|\nabla\phi|$ *if* $\phi = 2xz^4 - x^2y$ at (2, -2, -1).

Solution: The gradient of ϕ is $\nabla\phi = \vec{i}\,\dfrac{\partial\phi}{\partial x} + \vec{j}\,\dfrac{\partial\phi}{\partial y} + \vec{k}\,\dfrac{\partial\phi}{\partial z}$

$$\frac{\partial\phi}{\partial x} = 2z^4 - 2xy, \quad \frac{\partial\phi}{\partial y} = -x^2, \quad \frac{\partial\phi}{\partial z} = 8xz^3$$

$$\nabla\phi = \vec{i}\left(2z^4 - 2xy\right) + \vec{j}\left(-x^2\right) + \vec{k}\left(8xz^3\right)$$

$$[\nabla\phi]_{(2,-2,-1)} = \vec{i}\left(2(-1)^4 - 2(2)(-2)\right) + \vec{j}\left(-(2)^2\right) + \vec{k}\left(8(2)(-1)^3\right)$$

$$[\nabla\phi]_{(2,\,-2,\,-1)} = 10\vec{i} - 4\vec{j} - 16\vec{k}$$

$$\text{Magnitude of } \nabla\phi = \sqrt{10^2 + (-4)^2 + (-16)^2}$$

$$\therefore |\nabla\phi| = \sqrt{100 + 16 + 256} = \sqrt{372}.$$

2. Find the Directional derivative of $\phi = 3x^2 + 2y - 3z$ at (1, 1, 1) in the direction $2\vec{i} + 2\vec{j} - \vec{k}$.

Solution: The gradient of ϕ is $\nabla\phi = \vec{i}\,\dfrac{\partial\phi}{\partial x} + \vec{j}\,\dfrac{\partial\phi}{\partial y} + \vec{k}\,\dfrac{\partial\phi}{\partial z}$

$$\hat{n} = \left(\frac{2\vec{i} + 2\vec{j} - \vec{k}}{\sqrt{(2)^2 + (2)^2 + (-1)^2}}\right)$$ where $\hat{n}$ is the unit normal vector.

Directional derivative of ϕ is

$$\nabla\phi\cdot\hat{n} = \left[(6x\vec{i} + 2\vec{j} - 3\vec{k})\cdot\left(\frac{2\vec{i} + 2\vec{j} - \vec{k}}{3}\right)\right]_{(1,1,1)} \quad \therefore \nabla\phi\cdot\hat{n} = \frac{19}{3}.$$

3. Find the Unit normal vector to the surface $x^2 + y^2 - z = 1$ at the point (1,1 ,1). (APRIL /MAY 2017)

Solution:

Let $\phi = x^2 + y^2 - z - 1$

Now, The gradient of ϕ is $\nabla\phi = \vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z}$

$$\frac{\partial\phi}{\partial x} = 2x, \quad \frac{\partial\phi}{\partial y} = 2y, \quad \frac{\partial\phi}{\partial z} = -1$$

$$[\nabla\phi]_{(1,1,1)} = \left[(2x)\vec{i} + (2y)\vec{j} - \vec{k}\right]_{(1,1,1)} = 2\vec{i} + 2\vec{j} - \vec{k}$$

The unit normal vector is

$$\therefore \quad \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i} + 2\vec{j} - \vec{k}}{\sqrt{9}} = \frac{2\vec{i} + 2\vec{j} - \vec{k}}{3}$$

4. Find the angle between the surfaces $x \log z = y^2 - 1$ and $x^2y = 2 - z$ at the point (1, 1, 1)

Solution: Let $\phi_1 = y^2 - x\log z - 1$

$$\nabla\phi_1 = -\log z\,\vec{i} + 2y\vec{j} - \frac{x}{z}\vec{k}, \quad (\nabla\phi_1)_{(1,1,1)} = 2\vec{j} - \vec{k} \text{ and } |\nabla\phi_1| = \sqrt{5}$$

Let $\phi_2 = x^2y - 2 + z$

$$\nabla\phi_2 = \vec{i}(2xy) + \vec{j}x^2 + \vec{k}(1), \quad (\nabla\phi_2)_{(1,1,1)} = 2\vec{i} + \vec{j} + \vec{k} \text{ and } |\nabla\phi_2| = \sqrt{6}$$

$$\cos\theta = \frac{\nabla\phi_1 . \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|} = \frac{\left(2\vec{j} - \vec{k}\right).\left(2\vec{i} + \vec{j} + \vec{k}\right)}{\left(\sqrt{5}\right)\left(\sqrt{6}\right)} = \frac{0 + 2 - 1}{\sqrt{30}} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{\sqrt{30}}\right).$$

5. In what direction from (3, 1, -2) is the directional derivative of $\phi = x^2y^2z^4$ a maximum? Find the magnitude of this maximum. **(APR /MAY 2015)**

Solution: Given $\phi = x^2y^2z^4$

$$\nabla\phi = (2xy^2z^4)\vec{i} + (2yx^2z^4)\vec{j} + (4z^3x^2y^2)\vec{k}$$

$$[\nabla\phi]_{(3,\,1,\,-2)} = 96\vec{i} + 288\vec{j} - 288\vec{k}$$

$\therefore$ The maximum directional derivative occurs in the direction of $\nabla\phi = 96(\vec{i} + 3\vec{j} - 3\vec{k})$

The magnitude of this maximum directional derivative is $|\nabla\phi| = 96\sqrt{19}$.

6. Prove that Grad $\left(\frac{1}{r}\right) = \left(\frac{-\vec{r}}{r^3}\right)$ **(JAN 2016)**

Solution:

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

$\Rightarrow r^2 = x^2 + y^2 + z^2$

diff w.r.t x, we get

$2r\dfrac{\partial r}{\partial x} = 2x \Rightarrow \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

similarly $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$; $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

$$\text{grad}\left(\frac{1}{r}\right)=\left(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}\right)\left(\frac{1}{r}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right)=\frac{0-\frac{\partial r}{\partial x}}{r^2}=\frac{\frac{-x}{r}}{r^2}=\frac{-x}{r^3}$$

$$\frac{\partial}{\partial y}\left(\frac{1}{r}\right)=\frac{0-\frac{\partial r}{\partial y}}{r^2}=\frac{\frac{-y}{r}}{r^2}=\frac{-y}{r^3}$$

$$\frac{\partial}{\partial z}\left(\frac{1}{r}\right)=\frac{0-\frac{\partial r}{\partial z}}{r^2}=\frac{\frac{-z}{r}}{r^2}=\frac{-z}{r^3}$$

$$\therefore \text{grad}\left(\frac{1}{r}\right)=-\frac{x}{r^3}\vec{i}-\frac{y}{r^3}\vec{j}-\frac{z}{r^3}\vec{k}$$

$$=-\frac{1}{r^3}\left[x\vec{i}+y\vec{j}+z\vec{k}\right]$$

$$=-\frac{\vec{r}}{r^3}$$

7. **Evaluate** $\int_c \left(yz\vec{i}+xz\vec{j}+xy\vec{k}\right).d\vec{r}$ **where C is the boundary of a closed surface S.** **(JAN 2016)**

Solution: $curl\vec{F}=\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ yz & xz & xy\end{vmatrix}=\vec{i}(x-x)-\vec{j}(y-y)+\vec{k}(z-z)=\vec{0}$

$\vec{F}$ is irrotational $\Rightarrow$ $\vec{F}$ is conservative force $\Rightarrow \int_c \left(yz\vec{i}+xz\vec{j}+xy\vec{k}\right).d\vec{r}=0$

8. **Evaluate** $\nabla^2(\log r)$ **(MAY 2016)**

Solution :

$$\nabla^2(\log r)=\sum\frac{\partial^2}{\partial x^2}[\log r]=\sum\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\log r\right]$$

$$=\sum\frac{\partial}{\partial x}\left[\frac{1}{r}\frac{\partial r}{\partial x}\right]=\sum\frac{\partial}{\partial x}\left[\frac{1}{r}\frac{x}{r}\right]$$

$$=\sum\frac{\partial}{\partial x}\left[\frac{x}{r^2}\right]=\sum\left[\frac{r^2(1)-x\frac{\partial}{\partial x}(r^2)}{r^4}\right]=\sum\left[\frac{r^2-x(2r)\frac{\partial r}{\partial x}}{r^4}\right]$$

$$=\sum\left[\frac{r^2-x(2r)\frac{x}{r}}{r^4}\right]=\frac{3r^2(1)-x2r\frac{x}{r}-y2r\frac{y}{r}-z2r\frac{z}{r}}{r^4}$$

$$=\frac{3r^2-2x^2-2y^2-2z^2}{r^4}=\frac{3r^2-2r^2}{r^4}=\frac{1}{r^2}$$

9. **Find 'a', such that $\vec{F}=(3x-2y+z)\vec{i}+(4x+ay-z)\vec{j}+(x-y+2z)\vec{k}$ is solenoidal.** **(MAY 2015)**

Solution: We know that is $\vec{F}$ Solenoidal if $div\,\vec{F}=0$ *or* $\nabla.\vec{F}=0$

$$div\ \vec{F} = \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right).[(3x-2y+z)\vec{i}+(4x+ay-z)\vec{j}+(x-y+2z)\vec{k}] = 0$$

$$\Rightarrow \frac{\partial}{\partial x}(3x-2y+z)+\frac{\partial}{\partial y}(4x+ay-z)+\frac{\partial}{\partial z}(x-y+2z)=0$$

$$\Rightarrow 3 + a + 2 = 5 + a = 0 \quad \therefore\ a = -5.$$

10. If $\vec{A}$ and $\vec{B}$ are irrotational vectors, prove that $\vec{A}\times\vec{B}$ is solenoidal.

Solution: To prove $\vec{A}\times\vec{B}$ is solenoidal, we have to show that $\nabla\cdot(\vec{A}\times\vec{B}) = 0$

$\vec{A}$ is irrotational $\Rightarrow$ curl $\vec{A} = 0$ and $\vec{B}$ is irrotational $\Rightarrow$ curl $\vec{B} = 0$

$\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\text{curl}\,\vec{A}) - \vec{A}\cdot(\text{curl}\,\vec{B}) = \vec{B}\cdot 0 - \vec{A}\cdot 0 = 0$

$\therefore\ \vec{A}\times\vec{B}$ is solenoidal.

11. Show that the vector $\vec{F}=\left(6xy+z^3\right)\vec{i}+\left(3x^2-z\right)\vec{j}+\left(3xz^2-y\right)\vec{k}$ is irrotational.

Solution:

$$curl\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix} = \vec{i}(-1+1)-\vec{j}(3z^2-3z^2)+\vec{k}(6x-6x)=0$$

$\therefore\ \vec{F}$ is irrotational.

12. If $\vec{F} = 5xy\vec{i} + 2y\vec{j}$, evaluate $\int_C \vec{F}.d\vec{r}$ where C is the part of the curve $y = x^3$ between x =1 and x =2.

Solution: $y = x^3 \Rightarrow dy = 3x^2\,dx$

$$\int_c \vec{F}\cdot d\vec{r} = \int_c (5xydx + 2ydy) = \int_1^2 5x\left(x^3\right)dx + 2\left(x^3\right)3x^2dx$$

$$= \int_1^2 (5x^4+6x^5)dx = \left[x^5+x^6\right]_1^2$$

$$= 31 + 63 = 94.$$

13. If $\vec{F} = x^2\vec{i} + xy^2\vec{j}$, evaluate the line integral $\int_c \vec{F}\cdot d\vec{r}$ from (0,0) to (1,1) along the path $y = x$.

Solution:

$$\int_c \vec{F}\cdot d\vec{r} = \int_c x^2dx + xy^2dy \qquad (\because y = x)$$

$$= \int_0^1 (x^2+x^3)dx = \left[\frac{x^3}{3}+\frac{x^4}{4}\right]_0^1 = \frac{7}{12}.$$

14. If $\vec{F} = (4xy-3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^3z\ \vec{k}$. Check whether the integral $\int_C \vec{F}\cdot dr$ is independent of the path C.

Solution:

This integral is independent of the path of integration if $\nabla\times\vec{F}=0$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 4xy-3x^2z^2 & 2x^2 & -2x^3z\end{vmatrix} = \vec{i}(0-0)-\vec{j}(-6x^2z+6x^2z)+\vec{k}(4x-4x)=0$$

Hence the line integral is independent of path.

15. State Stoke's theorem. **(MAY 2016)**

Statement: If S is an open surface bounded by a simple closed curve C and if a vector function $\vec{F}$ is continuous and has continuous partial derivatives in S and on C, then $\iint_S curl\vec{F}\cdot\hat{n}\,ds = \int_C \vec{F}\cdot d\vec{r}$

where $\hat{n}$ is the outward unit normal vector to the surface.

16. State Green's Theorem.

Statement: If M(x ,y) and N(x ,y) are continuous function with continuous partial derivatives in a region R of the xy plane bounded by a simple closed curve C, then $\oint_C Mdx+Ndy=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$

where C is the curve described in positive direction.

17. State Gauss Divergence Theorem.

Statement: If V is the volume bounded by a closed surface S and if a vector function $\vec{F}$ is continuous partial derivative in V and on S , then $\iint_S \vec{F}\cdot\hat{n}\,ds=\iiint_V div\vec{F}\ dV$

where $\hat{n}$ is the outward unit normal vector to the surface.

18. Using Divergence theorem, evaluate $\iint_S xdydz+ydzdx+zdxdy$ over the surface of the sphere $x^2+y^2+z^2=a^2$

Solution: By Divergence theorem, $\iint_S \vec{F}\cdot\hat{n}\,ds=\iiint_V div\vec{F}\ dV$

$$\iint_S xdydz+ydzdx+zdxdy=\iiint_V \nabla.\vec{F}\ dv=\iiint_V 3\ dv$$

$$= 3\left(\frac{4}{3}\pi a^3\right) = 4\pi a^3.$$

19. Find the area of the circle of radius '*a*' using Green's theorem.

Solution: Green's theorem is $\int_C Mdx+Ndy=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$

By Greens theorem, we have, Area = $\iint_R dxdy=\frac{1}{2}\int_C(xdy-ydx)$

In a circle $x^2+y^2=a^2$, $x=a\cos\theta$ $y=a\sin\theta$, $\theta: 0\to 2\pi$

$$\therefore\ \text{Area} = \frac{1}{2}\int_0^{2\pi}(a^2\cos^2\theta+a^2\sin^2\theta)d\theta = \frac{a^2}{2}\int_0^{2\pi}d\theta = \pi a^2$$

20. If S is any closed surface enclosing a volume V and $\vec{F}=ax\vec{i}+by\vec{j}+cz\vec{k}$, Prove that $\iint_S \vec{F}\cdot\hat{n}\,ds=(a+b+c)V$

Solution: Gauss Divergence theorem is $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V div\vec{F}\ dv$ where div $\bar{F}$ = a+b+c

$$\iint_S \vec{F}\cdot\vec{n}\ ds = \iiint_V div\vec{F}\,(dV) = (a+b+c)\iiint_V dV = (a+b+c)V$$

PART B

1.(a) ***Given* : *div*(*gradr*^{n}) = ∇(∇*r*^{n}) = ∇^{2}*r*^{n}**

$$\nabla(r^n) = \vec{i}\,\frac{\partial}{\partial x}(r^n) + \vec{j}\,\frac{\partial}{\partial y}(r^n) + \vec{k}\,\frac{\partial}{\partial z}(r^n)$$

$$= \vec{i}nr^{n-1}\frac{x}{r} + \vec{j}nr^{n-1}\frac{y}{r} + \vec{k}nr^{n-1}\frac{z}{r}$$

$$= nr^{n-2}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$\therefore \nabla(r^n) = nr^{n-2}\vec{r}.$$

Now,

$$\nabla^2(r^n) = \nabla\cdot(nr^{n-2}\vec{r})$$

$$= n\nabla\cdot(r^{n-2}\vec{r})$$

$$= n\left[\nabla\left(r^{n-2}\right)\cdot\vec{r} + r^{n-2}\nabla\left(\vec{r}\right)\right]$$

$$= n\left[\left(n-2\right)r^{n-4}\vec{r}\cdot\vec{r} + 3r^{n-2}\right]$$

$$= n\left[\left(n-2\right)r^{n-4}r^2 + 3r^{n-2}\right]$$

$$= n\left(n+1\right)r^{n-2}$$

Put,

$$n = -1$$

$$\nabla^2\left(\frac{1}{r}\right) = \vec{0}$$

(b) ***Prove that curl*(*grad*φ) = 0, *using Stoke's theorem*** **(APR/MAY 2017)**

Solution : $\iint_s curl\vec{F}.ds = \int_c \vec{F}.dr$

Let , $\vec{F} = grad\phi$

$$\iint_s curl(grad\phi).ds = \int_c grad\phi.dr$$

$$= \int_c (\nabla\phi).d\vec{r}$$

$$= \int_c (\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}).(dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \int_c \frac{\partial\phi}{\partial x}.dx + \frac{\partial\phi}{\partial y}.dy + \frac{\partial\phi}{\partial z}.dz$$

$$= \int_c d\phi = 0$$

Since , for any open 2 sided surface S , provided it is bounded by the same simple closed curve C.
Hence , R.H.S :

$$\iint curl(grad\,\phi) = 0 \quad \text{(for any S, hence for any } d\vec{s}\text{)}.$$

2.(a) If $\nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$ find $\phi(x,y,z)$ given that $\phi(1,-2,2) = 4$. (MAY/JUN 2016)

Solution :

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z} \to (1)$$

$$Given: \nabla\varphi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k} \to (2)$$

$$\therefore comparing\ (1)\ and\ (2)$$

$$\frac{\partial\varphi}{\partial x} = 2xyz^3 \to (3)$$

$$\frac{\partial\varphi}{\partial y} = x^2z^3 \to (4)$$

$$\frac{\partial\varphi}{\partial z} = 3x^2yz^2 \to (5)$$

Integrating (3) w.r.t "x" (keeping y and z as constant)

$$\varphi = x^2yz^3 + f_1(y,z)$$

Integrating (4) w.r.t "y" (keeping x and z as constant)

$$\varphi = x^2yz^3 + f_2(x,z)$$

Integrating (5) w.r.t "z" (keeping x and y as constant)

$$\varphi = x^2yz^3 + f_3(x,y)$$

$\therefore\ \varphi = x^2yz^3 + c$,where c is constant.

$$Given: \varphi(1,-2,2) = 4$$

$$\therefore -16 + c = 4$$

$$\therefore c = 20$$

(b) Find 'a' and 'b' so that the surfaces $ax^3 - by^2z = (a+3)x^2$ and $4x^2y - z^3 = 11$ cut orthogonally at (2, -1, -3) (MAY / JUN 2016)

Solution :

$$Let\ \phi_1 = ax^3 - by^2z - (a+3)x^2$$

$$\phi_2 = 4x^2y - z^3 - 11$$

$$\nabla\phi_1 = \left[3ax^2 - (a+3)2x\right]\vec{i} - 2byz\vec{j} - by^2\vec{k}$$

$$\nabla\phi_2 = 8xy\vec{i} - 4x^2\vec{j} - 3z^2\vec{k}$$

$at\ (2,-1,-3)\nabla\phi_1 = (8a-12)\vec{i} - 6b\vec{j} - b\vec{k}$

$\nabla\phi_2 = 16\vec{i} - 16\vec{j} - 27\vec{k}$

Since the surfaces cut orthogonally at (2,-1,-3), $\nabla\phi_1 \cdot \nabla\phi_2 = 0$

$\Rightarrow -16(8a-12) - 16(6b) + 27b = 0$

$\Rightarrow -128a + 192 - 69b = 0$

$\Rightarrow 128a + 69b = 192 \rightarrow (1)$

Since the points (2,-1,-3) lies on the surface $\phi(x,y,z) = 0$ **,we have**

$8a + 3b - 4a = 12$

$\Rightarrow 4a + 3b = 12$

Solving equations (1) & (2) we get; $a = -2.333$, $b = 7.111$

3. (a) **Find the work done in moving a particle in the force field** $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$ **along the curve defined by** $x^2 = 4y$ **,** $3x^2 = 8z$ **from** $x = 0$ **to** $x = 2$**.** **(APR/MAY 2017)**

Given :

$$x^2 = 4y$$

$$\Rightarrow y = \frac{x^2}{4} \Rightarrow dy = \frac{2x}{4} = \frac{x}{2}$$

$$Also, 3x^2 = 8z$$

$$\Rightarrow z = \frac{3x^2}{8}$$

$$\Rightarrow dz = \frac{6x}{8} = \frac{3x}{4}$$

Work done = $\int_c \vec{F}.d\vec{r}$

$$= \int_c (3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}).(dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \int_c 3x^2dx + (2xz - y)dy + zdz$$

$$= \int_{x=0}^{2} 3x^2dx + (2x.\frac{3x^2}{8} - \frac{x^2}{4}).\frac{x}{2}.dx + \frac{3x^2}{8}.\frac{3x}{4}dx$$

$$= \int_{x=0}^{2} (3x^2 + \frac{3x^4}{8} - \frac{x^3}{8} + \frac{9x^3}{32})dx$$

$$=\left[\frac{3x^3}{3}+\frac{3x^5}{40}-\frac{x^4}{32}+\frac{9x^4}{128}\right]_0^2$$

$$=8+\frac{12}{5}-\frac{1}{2}+\frac{9}{8}$$

$$=\frac{320+96-20+45}{40}=\frac{441}{40}$$

Verify Green's theorem for $\int_C \left[x^2(1+y)dx+(x^3+y^3)dy\right]$ where C is the boundary of the region defined by the lines x = $\pm1$and y = ±1. **(MAY / JUN 2016)**

Solution :

(b)

$$Given: \int_c x^2(1+y)dx+\left(y^3+x^3\right)dy$$

$$M=x^2(1+y)$$

$$\frac{\partial M}{\partial y}=x^2$$

$$Also, N=y^3+x^3$$

$$\frac{\partial N}{\partial y}=3x^2$$

By Green's theorem ,

$$\int_c Mdx+Ndy=\iint_R\left(\frac{\partial N}{\partial y}-\frac{\partial M}{\partial y}\right)dxdy$$

Consider,

$$\iint_R\left(\frac{\partial N}{\partial y}-\frac{\partial M}{\partial y}\right)dxdy=\int_{-1}^{1}\int_{-1}^{1}\left(3x^2-x^2\right)$$

$$=\int_{-1}^{1}\int_{-1}^{1}\left(2x^2\right)dydx$$

$$=\int_{-1}^{1}2\left[\frac{x^3}{3}\right]_{-1}^{1}dy$$

$$=\int_{-1}^{1}\left[\frac{4}{3}\right]dy$$

$$=\frac{4}{3}[y]_{-1}^{1}=\frac{8}{3}\rightarrow(1)$$

consider

$$\int_c Mdx+Ndy=\int_{AB}+\int_{BC}+\int_{CD}+\int_{DA}$$

$$AlongAB, y=-1, dy=0andx\,\mathrm{var}\,iesfrom-1to1$$

$$\therefore\int_c Mdx+Ndy=\int_{-1}^{1}x^2(1-1)dx=0$$

$$AlongBC, x=1, dx=0$$

And y varies from -1 to 1,

$$\therefore \int_{BC} Mdx + Ndy = \int_{-1}^{1} \left(y^3 + 1\right)dy$$

$$= \left[\frac{y^4}{4} + 1\right]_{-1}^{1}$$

$$= 2$$

Along CD y=1,dy=0 and x varies from 1 to -1 ,

$$\therefore \int_{CD} Mdx + Ndy = \int_{1}^{-1} 2x^2 dx = \left[\frac{2x^3}{3}\right]_{1}^{-1} = \frac{-4}{3}$$

Along DA , x = -1,dx =1 and y varies from 1 to -1 ,

$$\therefore \int_{DA} Mdx + Ndy = \int_{1}^{-1} \left(y^3 - 1\right)dy = \left[\frac{y^4}{4} - y\right]_{1}^{-1} = 2$$

$$\int_{c} Mdx + Ndy = 0 + 2 - \frac{4}{3} + 2 = \frac{8}{3} \to (2)$$

$$\therefore (1) = (2)$$

Hence the theorem is verified .

4. Verify Gauss divergence theorem for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ taken over the cube bounded by the planes x = 0, x =1, y = 0, y =1, z = 0 and z = 1. (APR / MAY 2015)

Solution :G.D.T is $\iint_S \vec{F}.\hat{n}ds = \iiint_V \nabla.\vec{F}dv$

$$\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$$

$$\nabla.\vec{F} = 4z - 2y + y = 4z - y$$

Now

$$\iiint_V \nabla.\vec{F}dv = \int_0^1\int_0^1\int_0^1 (4z - y)dxdydz \; [\because dv = dxdydz]$$

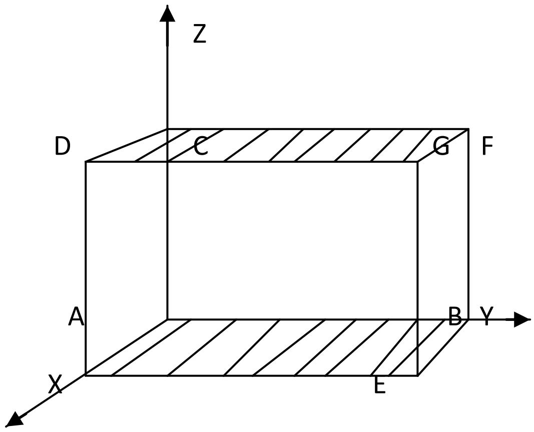
$$= \int_0^1\int_0^1 [4zx - yx]_0^1 \, dydz$$

$$= \int_0^1\int_0^1 (4z - y)dydz = \int_0^1 \left(4zy - \frac{y^2}{2}\right)_0^1 dz$$

$$= \int_0^1 \left(4z - \frac{1}{2}\right)dz = \left[4\frac{z^2}{2} - \frac{1}{2}z\right]_0^1$$

$$=\left(2-\frac{1}{2}\right)-0=\frac{3}{2}$$

Now $\int\int_S \vec{F}.\hat{n}ds = \iint_{s1} + \iint_{s2} + \iint_{s3} + \iint_{s4} + \iint_{s5} + \iint_{s6}$



(i) $\int\int_{S1} \vec{F}.\hat{n}ds$

$$= \iint_{AEGD} \left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right)\vec{i}\, dydz$$

$$= \iint_{AEGD} 4xzdydz$$

$$= \int_0^1\int_0^1 4zdydz \left[\because x=1 \text{ on } S_1\right]$$

$$= \int_0^1 \left(4yz\right)_0^1 dz = \int_0^1 4zdz$$

$$= \left[4\frac{z^2}{2}\right]_0^1 = \frac{4}{2} = 2$$

(ii) $\int\int_{S2} \vec{F}.\hat{n}ds$

$$= \iint_{OBFC} \left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right).(-i)dydz$$

$$= \int_0^1\int_0^1 -4xzdydz$$

$$= 0 \left[\because x=0 \text{ on } S_2\right]$$

(iii) $\int\int_{S3} \vec{F}.\hat{n}ds$

$$= \iint_{EBFG} \left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right)..jdxdz$$

$$= \int_0^1\int_0^1 -y^2dxdz \left[\because y=1 \text{ on } S_3\right]$$

$$= -\int_0^1\int_0^1 dxdz = -\int_0^1 (x)_0^1\, dz$$

$$= -\int_0^1 1dz = -(z)_0^1 = -1$$

(iv) $\int\int_{S4} \vec{F}.\hat{n}ds = \iint_{OADC} \left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right).(-J)dxdz$

$$= \iint_{OADC} y^2 dxdz$$

$= 0 \qquad [\because y = 0$ on this surface $]$

(v) $\int\int_{S5} \vec{F}.\hat{n}ds$

$$= \iint_{DEFC} \left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right).\vec{k}dxdy$$

$= \int_0^1\int_0^1 yzdxdy \; [\because z = 1$ on $S_5]$

$$= \int_0^1\int_0^1 ydxdy = \int_0^1 [yx]_0^1\, dy$$

$$= \int_0^1 ydy = \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}$$

(vi) $\int\int_{S6} \vec{F}.\hat{n}ds$

$$= \iint_{OAEB} \left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right).\left(-\vec{k}\right)dxdy$$

$= \int_0^1\int_0^1 -yzdxdy \; [\because z = 0$ on this surface $]$

$=0$

$$\int\int_S \vec{F}.\hat{n}ds = \iint_{s1} + \iint_{s2} + \iint_{s3} + \iint_{s4} + \iint_{s5} + \iint_{s6}$$

$$= \quad 2 + 0 - 1 + 0 + \frac{1}{2} + 0 = \frac{3}{2}$$

$$\therefore \int\int_S \vec{F}.\hat{n}ds = \int\int\int_V \nabla.\vec{F}dv$$

Hence G.D.T is verified

5 **Verify Stoke's theorem for the vector $\vec{F} = xy\vec{i} - 2yz\vec{j} - xz\vec{k}$, where S is the open surface of the rectangular parallelopiped formed by the planes x = 0, y = 0, x = 1, y = 2 and z = 3 above the XOY plane.**

Solution :

Stoke's theorem states that :

$$\int_C \vec{F}.d\vec{r} = \iint_S \nabla \times \vec{F}.d\vec{s}$$

$$\vec{F}.d\vec{r} = xydx - 2yzdy - xzdz$$

$$\int_C \vec{F}.d\vec{r} = \int_{OA} \vec{F}.d\vec{r} + \int_{AB} \vec{F}.d\vec{r} + \int_{BC} \vec{F}.d\vec{r} + \int_{CD} \vec{F}.d\vec{r}$$

Along OA y=0,dy=0,z=0,dz=0

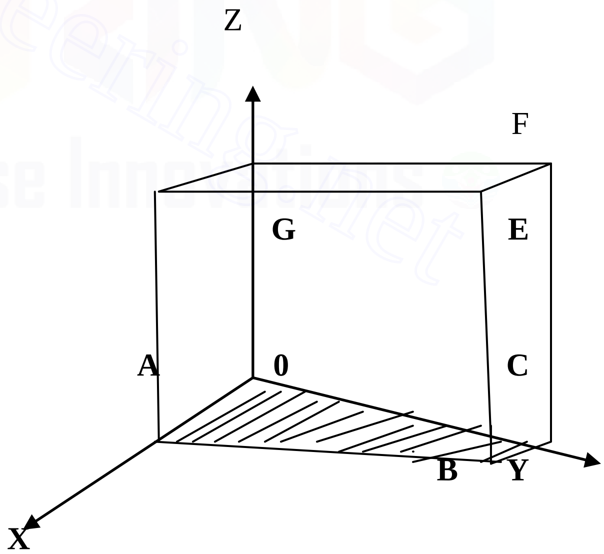
$$\int_{OA} \vec{F}.d\vec{r} = 0$$

Along AB ,x=1,dx=0,z=0,dz=0

$$\int_{AB} \vec{F}.d\vec{r} = 0$$

Along BC,y=2,dy=0,z=0,dz=0

$$\int_{BC} \vec{F}.d\vec{r} = \int_1^0 2xdx = -1$$



Along CO x=0,dx=0,z=0,dz=0

$$\int_{CO} \vec{F}.d\vec{r} = 0$$

$$\int_C \vec{F}.d\vec{r} = -1$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -2yz & -xz \end{vmatrix}$$

$$= \hat{i}(2y) - \hat{j}(-j) + \hat{k}(-x)$$

$= 2y\hat{i} + z\hat{j} - x\hat{k}$

$$\iint_S \nabla \times \vec{F} \cdot \hat{n}ds = \iint_{S1} + \iint_{S2} + ... + \iint_{S5}$$

$x = 0, \hat{n} = -\hat{i}$

$$\iint_S \nabla \times \vec{F}.\hat{n}ds = -\int_0^3\int_0^2 2ydydz = -\int_0^3 \left(y^2\right)_0^2 dz$$

$$= -4 \times 3 = -12$$

$x = 1, \hat{n} = \hat{i}$

$$\iint_{S_2} \nabla \times \vec{F} \cdot \hat{n}dS = -\int_0^3\int_0^2 2ydydz = 12$$

$y = 0, \hat{n} = -\hat{j}$

$$\iint_{S3} \nabla \times \vec{F} \cdot \hat{n}dS = -\int_0^3\int_0^1 zdxdz = -\int_0^3 zdz$$

$$= -\frac{9}{2}$$

$y = 2, \hat{n} = \hat{j}$

$$\iint_{S4} \nabla \times \vec{F} \cdot \hat{n}dS = \int_0^3\int_0^1 zdxdz = \frac{9}{2}$$

$z = 3\hat{n} = \hat{k}$

$$\iint_{S5} \nabla \times \vec{F} \cdot \hat{n}dS = -\int_0^2\int_0^1 xdxdy = -\int_0^2 \frac{1}{2}dy$$

= -1

$$\iint_S \nabla \times \vec{F}.\hat{n}ds = -12 + 12 - \frac{9}{2} + \frac{9}{2} - 1$$

= -1

L.H.S = R.H.S

6. Verify Gauss divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ taken over the rectangular parallelopiped bounded by the planes x = 0, x = a, y = 0, y = b, z = 0, and z = c.

(MAY / JUN 2014)

Solution:

By Gauss – Divergence theorem $\iint_S \vec{F} \cdot \vec{n} ds = \iiint_V div\vec{F} \cdot dV$

Evaluation of LHS:

$$\iint_S \vec{F} \cdot \vec{n} ds = \iint_{S_1} \vec{F} \cdot \hat{n}\, ds + \iint_{S_2} \vec{F} \cdot \vec{n} ds + \ldots + \iint_{S_6} \vec{F} \cdot \vec{n} ds$$

Over S_1: x = 0, $\hat{n} = -\vec{i}$

$$\iint_{S_1} \vec{F} \cdot \hat{n}\, ds = \int_0^c \int_0^b (yz)\, dy\, dz = \int_0^c \left[z\left(\frac{y^2}{2}\right)_0^b \right] dz = \frac{b^2}{2}\left(\frac{z^2}{2}\right)_0^c = \frac{b^2c^2}{4}$$

Over S_2: x = a, $\hat{n} = \vec{i}$

$$\iint_{S_2} \vec{F} \cdot \hat{n}\, ds = \int_0^c \int_0^b (-yz + a^2)\, dy\, dz = \int_0^b \left[-y\left(\frac{z^2}{2}\right)_0^c + a^2 [z]_0^c \right] dy$$

$$= -\frac{c^2}{2}\left(\frac{y^2}{2}\right)_0^b + ca^2 [y]_0^b = a^2bc - \frac{b^2c^2}{4}$$

Over S_3: y = 0, $\hat{n} = -\vec{j}$

$$\iint_{S_3} \vec{F} \cdot \hat{n}\, ds = \int_0^c \int_0^a (xz)\, dx\, dz = \int_0^c \left(\frac{x^2}{2} z\right)_0^a dz = \frac{a^2}{2}\left(\frac{c^2}{2}\right) = \frac{a^2c^2}{4}$$

Over S_4: y = b, $\hat{n} = \vec{j}$

$$\iint_{S_4} \vec{F} \cdot \hat{n}\, ds = \int_0^c \int_0^a (-xz + b^2)\, dx\, dz = \int_0^c \left[-z\left(\frac{a^2}{2}\right) + b^2 a \right] dz = ab^2c - \frac{a^2c^2}{4}$$

Over S_5: z = 0, $\hat{n} = -\vec{k}$

$$\iint_{S_5} \vec{F} \cdot \hat{n}\, ds = \int_0^b \int_0^a (xy)\, dx\, dy = \int_0^b \left[y\left(\frac{x^2}{2}\right)_0^a \right] dy = \frac{a^2b^2}{4}$$

Over S_6: $z = c$, $\hat{n} = \vec{k}$

$$\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \int_0^b\int_0^a (-xy + c^2)\,dx\,dy = \int_0^b \left[-y\left(\frac{a^2}{2}\right) + c^2 a\right] dy = abc^2 - \frac{a^2b^2}{4}$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \frac{b^2c^2}{4} + a^2bc - \frac{b^2c^2}{4} + \frac{a^2c^2}{4} + a\,b^2c - \frac{a^2c^2}{4} + \frac{a^2b^2}{4} + a\,bc^2 - \frac{a^2b^2}{4}$$

$$= a^2bc + ab^2c + abc^2 = abc(a+b+c)$$

Evaluation of RHS:

$$\nabla.\vec{F} = 2(x+y+z)$$

$$\iiint_V \nabla.\vec{F}\,dV = \int_0^c\int_0^b\int_0^a 2(x+y+z)\,dx\,dy\,dz$$

$$= 2\int_0^c\int_0^b \left[\frac{x^2}{2} + xy + xz\right]_0^a dy\,dz$$

$$= 2\int_0^c\int_0^b \left[\frac{a^2}{2} + ay + az\right] dy\,dz$$

$$= 2\int_0^c \left[\frac{a^2}{2}y + a\frac{y^2}{2} + ayz\right]_0^b dz$$

$$= 2\left[\frac{a^2bz}{2} + \frac{ab^2z}{2} + \frac{abz^2}{2}\right]_0^c$$

$$= 2\left[\frac{a^2bc}{2} + \frac{ab^2c}{2} + \frac{abc^2}{2}\right] = a^2bc + ab^2c + abc^2 = abc(a+b+c)$$

Hence, Gauss divergence theorem is verified.

UNIT-III ANALYTIC FUNCTIONS
PART-A

1. **Define an Analytic function (or) Holomorphic function (or) Regular function.**
   Solution: A function is said to be analytic at a point if its derivative exists not only at that point but also in some neighbourhood of that point.

2. **Define an Entire (or) an Integral function.**
   Solution: A function which is analytic everywhere in the finite plane except at $z = \infty$ is called an entire function.
   Example: e^z, sinz, coshz.

3. **State the necessary conditions for f(z) to be analytic.**

Solution: The necessary conditions for a complex function f(z) = u(x,y) + i v(x,y) to be analytic in a region R are

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad and \quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y} \quad \text{(i.e) C – R equations.}$$

4. State the Sufficient conditions for f (z) to be analytic.

Solution:

If the partial derivatives $\frac{\partial u}{\partial x},\frac{\partial v}{\partial y},\frac{\partial v}{\partial x},\frac{\partial u}{\partial y}$ exist and continuous in D and satisfies the conditions $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ *and* $\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$. Then the function f(z) is analytic in a domain D.

5. Give an example of a function where u and v are harmonic but u + iv is not analytic. (MAY/JUNE 2016)

Solution:

Let $u = x$ and $v = -y$

$u_x = 1, u_y = 0, u_{xx} = 0, u_{yy} = 0$;

$v_x = 0, \quad v_y = -1$

$v_{xx} = 0, \quad v_{yy} = 0$

$u_{xx} + u_{yy} = 0 \quad \Rightarrow u$ is harmonic.

and $v_{xx} + v_{yy} = 0 \quad \Rightarrow v$ is harmonic.

But $u_x \neq v_y$ and $u_y \neq v_x \Rightarrow u$ and v are not analytic.

6. Find the critical points of the transformation $w^2 = (z-\alpha)(z-\beta)$ (MAY/JUNE 2016)

Solution: Given: $w^2 = (z-\alpha)(z-\beta) - - - - - - - -(1)$

Critical points : If w = f (z) then $\frac{dw}{dz}=0$ *and* $\frac{dz}{dw}=0$

Differentiate with respect to z, we get

$$2w\frac{dw}{dz}=(z-\alpha)+(z-\beta)=2z-(\alpha+\beta)- - - - - - - - -(2)$$

$$=> \frac{dw}{dz}=\frac{2z-(\alpha+\beta)}{2w}- - - - - - - - - - - - - - - - -(3)$$

$$\frac{dw}{dz}=0 => z=\frac{\alpha+\beta}{2} \quad and \quad \frac{dz}{dw}=0 => \frac{2w}{2z-(\alpha+\beta)}=0$$

$$=> w=0 \ => (z-\alpha)(z-\beta)=0 \ => z=\alpha,\beta$$

The critical points are $\frac{\alpha+\beta}{2},\alpha$ *and* β

7. Define Harmonic function.

Solution:

Any function which possess continuous second order partial derivatives and which satisfies Laplace equation is called a harmonic function. (i.e) If $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$, then f is harmonic then

8. The real part of an analytic function f(z) is constant, prove that f(z) is a constant function. (MAY 2017)

Solution:

Let f(z) = u + iv

Given u = constant. $\Rightarrow u_x = 0$ and $u_y = 0$

by C-R equations, $u_x = 0 \Rightarrow v_y = 0$ and $u_y = 0 \Rightarrow v_x = 0$

$f'(z) = u_x + iv_x = 0 + i0 = 0$

Integrating, $f(z) = c$ (where c is a constant)

9. Define Conformal transformation.

Solution: A mapping or transformation which preserves angles in magnitude and in direction between every pair of curves through a point is said to be conformal transformation.

10. Define Isogonal transformation.

Solution: A transformation under which angles between every pair of curves through a point are preserved in magnitude but altered in sense is said to be isogonal at that point.

11. Define Bilinear transformation (or) Mobius transformation (or) linear fractional transformation.

Solution: The transformation $w = \frac{az+b}{cz+d}$, $ad - bc \neq 0$ where a, b,c,d are complex numbers is called a bilinear transformation. This is also called as Mobius or linear fractional transformation.

12. Define Cross Ratio.

Solution: The cross ratio of four points z_1, z_2, z_3, z_4 is given by $\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$.

13. Show that f (z) = $|z|^2$ is differentiable at z = 0 but not analytic at z = 0. **(APR / MAY 2015)**

Solution: Let z = x + iy and $\bar{z} = x - iy$

$$|z|^2 = z\bar{z} = x^2 + y^2$$

$$f(z) = |z|^2 = (x^2 + y^2) + i0$$

$$u = x^2 + y^2 \quad , \quad v = 0$$

$$u_x = 2x \quad , \quad v_x = 0$$

$$u_y = 2y \quad , \quad v_y = 0$$

So the C-R equations $u_x = v_y$ and $u_y = -v_x$ are not satisfied everywhere except at z = 0.

So f (z) may be differentiable only at z = 0. Now $u_x = 2x$, $v_y = 0$ and $u_y = 2y$, $v_x = 0$ are continuous everywhere and in particular at (0, 0). So f (z) is differentiable at z = 0 only and not analytic.

14. Determine whether the function $\bar{z}$ is analytic or not. **(MAY / JUN 2014)**

Solution: Let z = x + iy

$\bar{z} = x - iy$

u = x , v = - y

$u_x = 1 \qquad v_x = 0$

$u_y = 0 \qquad v_y = -1$

$u_x \neq v_y \qquad v_x = -u_y$

C-R equations are not satisfied. Therefore f (z) is not analytic.

15. Find the map of the circle $|z| = 3$ under the transformation w = 2z **(NOV / DEC 2012)**

Solution: Given w = 2z , $|z| = 3$

$$|w| = 2|z|$$

$$|w| = 2(3) = 6$$

Hence the image of the circle $|z| = 3$ in the z-plane maps to the circle $|w| = 6$ in the w-plane.

16. Show that the function u = 2x – x^3 + $3xy^2$ is harmonic.

Solution: Given u = 2x – x^3 + $3xy^2$

$$u_x = 2 - 3x^2 + 3y^2 \qquad u_y = 6xy$$

$$u_{xx} = -6x \qquad u_{yy} = 6x$$

$$u_{xx} + u_{yy} = -6x + 6x = 0$$

Hence u is harmonic

17. Find a function w such that w = u + iv is analytic, if u = e^x siny

Solution: Given u = e^x siny

$$u_x = e^x \sin y \qquad u_y = e^x \cos y$$

$$u_x(z,0) = e^z(0) = 0 \qquad u_y(z,0) = e^z(1) = e^z$$

By Milne Thomson's method

$$f(z) = \int u_x(z,0)dz - i\int u_y(z,0)dz = 0 - i\int e^z dz = -ie^z + C$$

18. Find the image of the hyperbola $x^2 - y^2 = 10$ under the transformation w = z^2

Solution: $w = z^2 \Rightarrow u + iv = (x+iy)^2 = x^2 - y^2 + i2xy$

$u = x^2 - y^2$ and v = 2xy ; $x^2 - y^2 = 10$ (i.e) u = 10

Hence the image of the hyperbola $x^2 - y^2 = 10$ under the transformation w = z^2 is u = 10 which is a straight line in w plane.

19. Obtain the invariant points of the transformation $w = \frac{z-1}{z+1}$ (APR / MAY 2015)

Solution: Given: $w = \frac{z-1}{z+1}$

The invariant points are obtained by replacing w by z.

$$\text{i.e, } z = \frac{z-1}{z+1} \Rightarrow z^2 + 1 = 0 \therefore z = \pm i$$

20. Find the image of the circle $|z| = 1$ by the transformation w = z + 2 + 4i

Solution:

Given: w = z + 2 + 4i

u + iv = x + iy + 2 + 4i = (x + 2) + i (y + 4)

u = x + 2, v = y + 4

⇨ x = u – 2, y = v – 4

⇨ $|z| = 1$

$x^2 + y^2 = 1$ Hence $(u – 2)^2 + (v – 4)^2 = 1.$

$\therefore$ The circle in the z-plane is mapped into the circle in the w -plane with centre (2, 4) and radius 1.

PART B

1. If $f(z)$ is a regular function of z, prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) log|f(z)|=0$ (APR / MAY 2017)

Solution :

Given

$$f(z)=u+iv$$

$$|f(z)|=\sqrt{u^2+v^2}$$

$$\log|f(z)|=\frac{1}{2}\log(u^2+v^2)$$

$$\nabla^2\log|f(z)|=\nabla^2\frac{1}{2}\log(u^2+v^2)$$

$$=\frac{1}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log(u^2+v^2)$$

$$=\frac{1}{2}\frac{\partial^2}{\partial x^2}[\log(u^2+v^2)]+\frac{1}{2}\frac{\partial^2}{\partial y^2}[\log(u^2+v^2)] \qquad -----(1)$$

consider,

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}[\log(u^2+v^2)]=\frac{1}{2}\frac{\partial}{\partial x}\left[\frac{1}{u^2+v^2}\left(2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}\right)\right]$$

$$=\frac{(u^2+v^2)[uu_x+vv_x+u_x^2+v_y^2]-[uu_y+vv_y]^2}{(u^2+v^2)^2}$$

$$\textit{similarly}\quad \frac{1}{2}\frac{\partial^2}{\partial y^2}[\log(u^2+v^2)]=\frac{(u^2+v^2)[uu_y+vv_y+u_y^2+v_x^2]-[uu_y+vv_y]^2}{(u^2+v^2)^2}$$

Substitute in equation (1), we get $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) log|f(z)|=0$

2. Prove that $u=x^2-y^2$ & $v=\frac{-y}{x^2+y^2}$ are harmonic functions but not harmonic conjugate. (NOV 2014)

Solution :

$$u_x=2x;\quad u_y=-2y$$

$$u_{xx}=2\quad;\quad u_{yy}=-2$$

$$u_{xx}+u_{yy}=0$$

similarly

$$v_x = \frac{2xy}{(x^2+y^2)^2} \quad and \quad v_y = \frac{(y^2-x^2)}{(x^2+y^2)^2}$$

$$v_{xx} + v_{yy} = 0$$

Hence u and v are harmonic.

But C-R Equations are not satisfied

Hence u+iv is not an analytic function

3. **Prove that $w = \frac{z}{1-z}$ maps the upper half of the z-plane into the upper half of the w-plane. What is the image of the circle $|z| = 1$ under this transformation? (NOV / DEC 2012)**

Solution :

$$w = \frac{z}{1-z} \Rightarrow w(1-z) = z$$

$$w - wz = z \qquad w = (w+1)z \qquad w = (w+1)z$$

$$z = \frac{w}{w+1}$$

Put $z = x+iy, \quad w = u+iv$

$$x+iy = \frac{u+iv}{u+iv+1} \qquad = \frac{(u+iv)(u+1)-iv}{(u+iv+1)(u+1)-iv}$$

$$= \frac{u(u+1)-iuv+iv(u+1)+v^2}{(u+1)^2+v^2} \qquad = \frac{(u^2+v^2+u)+iv}{(u+1)^2+v^2}$$

Equating real and imaginary parts

$$x = \frac{u^2+v^2+u}{(u+1)^2+v^2}, \; y = \frac{v}{(u+1)^2+v^2}$$

$$y = 0 \Rightarrow \frac{v}{(u+1)^2+v^2} = 0$$

$$y > 0 \Rightarrow \frac{v}{(u+1)^2+v^2} > 0 \Rightarrow v > 0$$

Thus the upper half of the z plane is mapped onto the upper half of the w plane.

4. **Prove that the real and imaginary parts of an analytic function are harmonic function. (MAY 2014)**

Proof:

Let f (z) = u + iv be an analytic function of z. Then by C- R equations we have,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{..........(1)} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \text{.........(2)}$$

Differentiating (1) partially with respect to x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \text{..........(3)}$$

Differentiating (2) partially with respect to y, we get

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial y \partial x} \text{..........(4)}$$

Adding (3) and (4), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

$\therefore$ u satisfies the Laplace equation.

Similarly

Differentiating (1) partially with respect to y, we get

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x} \text{..........(5)}$$

Differentiating (2) partially with respect to x, we get

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} \text{..........(6)}$$

Adding (5) and (6), we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x} = 0$$

$\therefore$ v satisfies the Laplace equation.

5. **Show that the transformation $w = \frac{1}{z}$ transforms in general, circles and straight lines into circles and straight lines.** **(APR / MAY 2017)**

Solution :

$$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$

$$\Rightarrow x + iy = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$$

$$\Rightarrow x = \frac{u}{u^2 + v^2} \text{ and } y = \frac{-v}{u^2 + v^2}$$

Consider the equation $a(x^2+y^2)+bx+cy+d=0$ ------------(1)

This equation represents a circle if $a \neq 0$ and a straight line if $a = 0$

Under the transformation $w = \frac{1}{z}$ equation (1) becomes

$$d(u^2+v^2)+bu - cv + a = 0 \text{---------------(2)}$$

This equation represents a circle if $d \neq 0$ and a straight line if $d = 0$

Value of a and d	Equation (1) and (2)	Conclusion
$a \neq 0,\ d \neq 0$	Equation (1) and (2) represents a circle, not passing through the origin, in the z–plane and w–plane	The transformation maps a circle not passing through the origin in z–plane into a circle not passing through the origin in w–plane
$a \neq 0,\ d = 0$	Equation (1) represents a circle passing through the origin in the z–plane and equation (2) represents a straight line not passing through the origin in w–plane	The transformation maps a circle passing through the origin in z–plane into a straight line not passing through the origin in w–plane
$a = 0,\ d \neq 0$	Equation (1) represents a straight line not passing through the origin in the z–plane and equation (2) represents a circle passing through the origin in w–plane	The transformation maps a straight line not passing through the origin in the z–plane into a circle passing through the origin in w–plane
$a = 0,\ d = 0$	Equation (1) and (2) represents a straight line passing through the origin in the z–plane and w–plane	The transformation maps represents a straight line passing through the origin in z–plane into a straight line passing through the origin in w–plane

Thus the transformation $w = \frac{1}{z}$ maps the totality of circles and straight lines as circles and straight lines.

6. Find the bilinear transformation which maps the points of z-plane -1, 0, 1 into the points -1, -i, 1 of w-plane respectively. **(APR / MAY 2017)**

Solution :

Cross-ratio

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-(-1))(-i-1)}{(w-1)(-i-(-1))} = \frac{[z-(-1)][0-(-1)]}{[z-1][0-(-1)]}$$

$$\therefore\ \frac{w+1}{w-1} = \frac{1+z}{1-z}\left(\frac{i-1}{i+1}\right)$$

$$w = \frac{z-i}{1-iz}$$

7. Can $v = \tan^{-1}\left(\frac{y}{x}\right)$ be the imaginary part of an analytic function? If so construct an analytic function f(z) = u + iv, taking v as the imaginary part and hence find u. **(MAY / JUNE 2016)**

Solution :

Given $\mathbf{v = \tan^{-1}\left(\frac{y}{x}\right)}$

$$v_x = \frac{-y}{x^2 + y^2} \quad and \quad v_y = \frac{x}{x^2 + y^2}$$

By Milne Thomson Rule

$$f(z) = \int v_y(z,0)dz + i\int v_x(z,0)dz + C \qquad , C - complex\,const.$$

$$f(z) = \log z + C$$

To find u:

$$f(z) = \log(re^{i\theta}) \quad [\because z = re^{i\theta}]$$

$$u + iv = \log r + i\theta$$

$$u = \log r = \frac{1}{2}\log(x^2 + y^2)$$

8. **Prove that $\mathbf{w = \frac{z}{z+a}}$ where $\mathbf{a \neq 0}$ is analytic whereas $\mathbf{w = \frac{\bar{z}}{\bar{z}+a}}$ is not analytic. (MAY/ JUNE 2016)**

Solution:

$$w = \frac{z}{z+a} = \frac{x+iy}{x+iy+a} = \frac{x+iy}{(x+a)+iy} = \frac{x+iy}{(x+a)+iy}\left(\frac{(x+a)-iy}{(x+a)-iy}\right)$$

$$= \frac{(x+iy)((x+a)-iy)}{(x+a)^2+y^2} = \frac{x(x+a)+y^2}{(x+a)^2+y^2} + i\frac{(x+a)y-xy}{(x+a)^2+y^2}$$

$$w = \underbrace{\frac{x(x+a)+y^2}{(x+a)^2+y^2}}_{u} + i\underbrace{\frac{ay}{(x+a)^2+y^2}}_{v}$$

$$u = \frac{x(x+a)+y^2}{(x+a)^2+y^2};$$

$$u_x = \frac{((x+a)^2+y^2)(2x+a) - (x(x+a)+y^2)(2(x+a))}{((x+a)^2+y^2)^2}$$

$$= \frac{2x(x+a)+2xy^2-2x^2(x+a)-2xy^2-2ax(x+a)-2ay^2}{((x+a)^2+y^2)^2}$$

$$u_x = \frac{a((x+a)^2-y^2)}{((x+a)^2+y^2)^2} \ldots(1)$$

$$u_y = \frac{((x+a)^2+y^2)(2y)-(x(x+a)+y^2)(2y)}{((x+a)^2+y^2)^2}$$

$$= \frac{2y((x+a)^2+y^2-(x(x+a)+y^2))}{((x+a)^2+y^2)^2}$$

$$= \frac{2y(x^2+ax+a^2+y^2-x^2-ax-y^2)}{((x+a)^2+y^2)^2}$$

$$u_y = \frac{2ay(x+a)}{((x+a)^2+y^2)^2} \ldots (2)$$

$$v = \frac{ay}{(x+a)^2+y^2};$$

$$v_x = \frac{((x+a)^2+y^2)(0)-(ay)(2(x+a))}{((x+a)^2+y^2)^2}$$

$$v_x = \frac{-2ay(x+a)}{((x+a)^2+y^2)^2} \ldots (3)$$

$$v_y = \frac{((x+a)^2+y^2)(a)-(ay)(2y)}{((x+a)^2+y^2)^2}$$

$$= \frac{a((x+a)^2+y^2-2y^2)}{((x+a)^2+y^2)^2}$$

$$v_y = \frac{a((x+a)^2-y^2)}{((x+a)^2+y^2)^2} \ldots (4)$$

From (1) and (4), $u_x = v_y$

From (2) and (3), $u_y = -v_x$

Also u_x, u_y, v_x, v_y are continuous functions in x and y.

Hence $w = \frac{z}{z+a}$ is analytic.

$$\text{Now } w = \frac{\bar{z}}{\bar{z}+a} = \frac{x-iy}{x-iy+a} = \frac{x-iy}{(x+a)-iy} = \frac{x-iy}{(x+a)-iy}\left(\frac{(x+a)+iy}{(x+a)+iy}\right)$$

$$= \frac{(x-iy)((x+a)+iy)}{(x+a)^2+y^2} = \frac{x(x+a)+y^2}{(x+a)^2+y^2} + i\frac{(-(x+a)y+xy)}{(x+a)^2+y^2}$$

$$w = \underbrace{\frac{x(x+a)+y^2}{(x+a)^2+y^2}}_{u} + i\underbrace{\frac{-ay}{(x+a)^2+y^2}}_{v}$$

$$u = \frac{x(x+a)+y^2}{(x+a)^2+y^2};$$

$$u_x = \frac{a((x+a)^2-y^2)}{((x+a)^2+y^2)^2}\ldots(5)$$

$$u_y = \frac{2ay(x+a)}{((x+a)^2+y^2)^2}\ldots(6)$$

$$v = \frac{-ay}{(x+a)^2+y^2};$$

$$v_x = \frac{2ay(x+a)}{((x+a)^2+y^2)^2}\ldots(7)$$

$$v_y = \frac{-a((x+a)^2-y^2)}{((x+a)^2+y^2)^2}\ldots(8)$$

From (5) and (8), $u_x \neq v_y$

From (6) and (7), $u_y \neq -v_x$

Hence $w = \dfrac{\bar{z}}{\bar{z}+a}$ is not analytic.

9. **Find the bilinear transformation that transforms the points z = 1, i, -1 of the z-plane into the points w = 2, i, -2 of the w-plane. (MAY/ JUNE 2016)**

Solution :

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-2)(i+2)}{(w+2)(i-2)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$w = \frac{z(6+18i)-2i+6}{z(-i+3)+3+9i}$$

$$w = \frac{z(6+18i)+(6-2i)}{z(-i+3)+(3+9i)}$$

10. **Determine the analytic function f(z) = u + iv, given that 2u+3v = e^x (cos y – sin y).**

(APR / MAY 2017)

Solution:

$$\text{Given } 2u+3v = e^x\left[\cos y - \sin y\right]$$

$$f(z) = u+iv \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

$$3if(z) = 3iu-3v \qquad \ldots\ldots\ldots\ldots\ldots(2)$$

$(1)\times 2 \Rightarrow 2f(z) = 2u + i2v$ ....(3)

$(3)-(2) \Rightarrow (2-3i)f(z) = (2u+3v)+i(2v-3u)$ .....(4)

$F(z) = U + iV$

$\therefore 2u+3v = U = e^x[\cos y - \sin y]$

$\phi_1(x,y) = \frac{\partial U}{\partial x} = e^x \cos y - e^x \sin y$

$\phi_1(z,o) = e^z$

$\phi_2(x,y) = \frac{\partial u}{\partial x} = -e^x \sin y - e^x \cos y$

$\phi_2(z,o) = -e^z$

By Milne Thomson method

$F'(z) = \phi_1(z,o) - i\phi_2(z,o)$

$\int F'(z)\, dz = \int e^z dz - i\int -e^z dz$

$F(z) = (1+i)e^z + C - - - - - (5)$

From (4) & (5)

$(1+i)e^z + C = (2-3i)f(z)$

$f(z) = \frac{1+i}{2-3i}e^z + \frac{C}{2-3i}$

$f(z) = \frac{-1+5i}{13}e^z + \frac{C}{2-3i}$

UNIT – IV COMPLEX INTEGRATION

PART – A

1. State Cauchy's Integral Theorem. **(APR / MAY 2015)**

Solution: If $f(z)$ is analytic at every point of the region R bounded by a simple closed curve C and if $f'(z)$ is continuous at all points inside and on C, then $\int_C f(z)\,dz = 0$

2. What is the value of the integral $\int_C \frac{3z^2+7z+1}{z+1}dz$, where C is $|z| = \frac{1}{2}$? **(NOV / DEC 2014)**

Solution: Let $f(z) = \int_C \frac{3z^2+7z+1}{z+1}dz$

Singular point is given by z + 1 = 0 $\Rightarrow$ z = –1

Put z = –1 $\therefore |z| = |-1| = 1 > \frac{1}{2}$

z = –1 lies outside C. $\therefore f(z)$ is analytic inside and on C, $f'(z)$ is continuous inside C

Hence by Cauchy's integral theorem $\int_C f(z)dz = 0 \Rightarrow \int_C \frac{3z^2+7z+1}{z+1}dz = 0$

3. State Cauchy's integral formula for nth derivative.

Solution: If $f(z)$ is analytic inside and on a simple closed curve C and z = a is any interior point of the region R enclosed by C, then $f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$

4. Evaluate $\int_C \frac{dz}{2z+3}$ **if C is | z | = 2.**

Solution: The pole $z = \frac{-3}{2}$ lies inside the circle **| z | = 2** $\therefore \int_C \frac{dz}{2z+3} = 2\pi i \times R$.

Hence $\int_C \frac{dz}{2z+3} = \frac{1}{2}\int_C \frac{dz}{z+\frac{3}{2}} = \frac{1}{2}\int_C \frac{dz}{z-\left(-\frac{3}{2}\right)} = \frac{1}{2}\times 2\pi i f\left(\frac{-3}{2}\right)$ $\quad \because \begin{bmatrix} f(z)=1 \\ f\left(-\frac{3}{2}\right)=1 \end{bmatrix}$

$= \pi i$

5. Evaluate $\int_C \frac{\cos \pi z}{z-1}dz$ **if C is | z | = 2.**

Solution: Let I = $\int_C \frac{\cos \pi z}{z-1}dz$. Singular point is given by z – 1 = 0 ⇒ z = 1

Let f(z) = cos πz which is analytic inside and on C.

$\therefore$ I = $\int_C \frac{\cos \pi z}{z-1}dz \Rightarrow$ I = $\int_C \frac{f(z)}{z-1}dz = 2\pi i\, f(1) = 2\pi i \cos \pi = -2\pi i$.

6. Expand $\frac{1}{z^2}$ **at z = 2 as a Taylor's series.** **(MAY / JUNE 2016)**

Solution: Let $f(z) = \frac{1}{z^2}$. z = 2 is a regular point so, we can find Taylor's series about z = 2.

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \ldots$$

$f(z) = \frac{1}{z^2}$ $\qquad f(2) = \frac{1}{4}$

$f'(z) = -\frac{2}{z^3}$ $\qquad f'(2) = -\frac{1}{2^2}$

$f''(z) = \frac{6}{z^4}$ $\qquad f''(2) = \frac{3}{2^3}$

$$f(z) = f(2) + \frac{f'(2)}{1!}(z-2) + \frac{f''(2)}{2!}(z-2)^2 + \ldots$$

$$= \frac{1}{4} - \frac{1}{4}(z-2) + \frac{3}{8}\frac{1}{2!}(z-2)^2 + \ldots = \frac{1}{4}\left[3 - z + \frac{3(z-2)^2}{2} + \ldots\right]$$

7. Obtain the Taylor's series expansion of $\log(1+z)$ **when z = 0.**

Solution: Let $f(z) = \log(1+z)$ $\qquad f(0) = \log 1 = 0$

$f'(z) = \frac{1}{1+z}$ $\qquad f'(0) = \frac{1}{1+0} = 1$

$f''(z) = \frac{-1}{(1+z)^2}$ $\qquad f''(0) = -1$

$f'''(z) = \frac{2}{(1+z)^3}$ $\qquad f'''(0) = 2$

$f^{iv}(z) = \frac{-6}{(1+z)^4}$ $\quad f^{iv}(0) = -6$

$$\log(1+z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + ... = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + ...$$

8. Obtain the Laurent expansion of the function $\frac{e^z}{(z-1)^2}$ in the neighbourhood of its singular point. Hence find the residue at that point.

Solution: $z = 1$ is a pole of order 2. Put

$$z - 1 = u \therefore f(z) = \frac{e^z}{(z-1)^2} = \frac{e.e^u}{u^2} = \frac{e}{u^2}\left[1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + ...\right] = \frac{e}{u^2} + \frac{e}{u} + \frac{e}{u^2}\left[\frac{u^2}{2!} + \frac{u^3}{3!} + ...\right]$$

$\therefore$ Residue of f(z) = coefficient of $\frac{1}{u} = e$

9. State Laurent's series.

Solution: If C_1, C_2 are two concentric circles with centre at z = a and radii r_1 and r_2 ($r_1 < r_2$) and if f(z) is analytic inside and on the circles and within the annular region between C_1 and C_2, then for any z in the annular region, we have $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$,where

$$a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{(z-a)^{n+1}}dz \quad \text{and} \quad b_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{(z-a)^{-n+1}}dz$$

10. State Cauchy's Residue theorem. **(MAY / JUN 2014)**

Statement: If f(z) is analytic inside a closed curve C except at a finite number of isolated singular points $a_1, a_2, \ldots a_n$ inside C, then $\int_C f(z)dz = 2\pi i \times$(sum of the residues of f(z) at the singular points lying inside C).

11. Determine the poles and residues at each pole of the function f(z) = cot z

Solution: Given f(z) = cot z = $\frac{\cos z}{\sin z}$

The poles of f(z) are given by sin z = 0 $\Rightarrow$ z = nπ when n = 0, ±1, ±2,...

Residue of f(z) at z = nπ is $\frac{P(n\pi)}{Q'(n\pi)} = \left(\frac{\cos z}{\frac{d}{dz}(\sin z)}\right)_{z=n\pi} = \frac{\cos n\pi}{\cos n\pi} = 1$

12. Find the residue of f(z) = $z^2 \sin\left(\frac{1}{z}\right)$ at z = 0

Solution: Given f(z) = $z^2 \sin\left(\frac{1}{z}\right) = z^2\left[\frac{\frac{1}{z}}{1!} - \frac{\left(\frac{1}{z}\right)^3}{3!} + ...\right] = \frac{z}{1} - \frac{1}{6z} + ...$

Residue of f(z) = $z^2 \sin\left(\frac{1}{z}\right)$ at z = 0 is the coefficient of $\frac{1}{z} = \frac{-1}{6}$

13. Discuss the nature of singularities of $\frac{\sin z - z}{z^3}$ at $z = 0$

Solution: Let $f(z) = \frac{\sin z - z}{z^3}$. The function $f(z)$ is not defined at $z = 0$. But by L'Hospital's rule we have,

$$\lim_{z\to 0}\frac{\sin z - z}{z^3} = \lim_{z\to 0}\frac{\cos z - 1}{3z^2} = \lim_{z\to 0}\frac{-\sin z}{6z} = \lim_{z\to 0}\frac{-\cos z}{6} = \frac{-1}{6}$$

Since the limit exists and is finite, the singularity at $z = 0$ is a removable singularity.

14. Evaluate $\int_C \frac{e^{2z}}{(z+2)^4}dz$ where C is $|z| = 1$ using Cauchy's integral formula.

Solution: :

By Cauchy Residue Theorem, $f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$

$Z = -2$ lies out of the given circle $|z| = 1$, hence $\int_C \frac{e^{2z}}{(z+2)^4}dz = 0$

15. If f (z) = $-\frac{1}{z-1} - 2[1 + (z-1) + (z-1)^2 + ...]$, find the residue of f (z) at z = 1. (NOV/DEC 2012)

Solution: Given the series expansion of f(z) about z = 1 is = $-\frac{1}{z-1} - 2[1 + (z-1) + (z-1)^2 + ...]$

Residue at z = 1 is the coefficient of $(z-1)^{-1}$

$\therefore$ Residue of f(z) at z = 1 is –1

16. Find the value of $\int_C \frac{4}{(z-4)^3(z-2)}dz$ where C is | z | < 3

Solution: Given $f(z) = \frac{4}{(z-4)^3(z-2)}$,

The poles are given by $(z-4)^3(z-2) = 0 \Rightarrow z = 2$ & z = 4 (thrice)

$z = 2$ is a simple pole and lies inside C. But z = 4 is a pole of order 3 and lies outside C.

$$[\text{Res of f(z)}]_{z=2} = \lim_{z\to 2}(z-2)f(z) = \lim_{z\to 2}(z-2)\frac{4}{(z-4)^3(z-2)} = \lim_{z\to 2}\frac{4}{(z-4)^3} = \frac{4}{(-2)^3} = \frac{-1}{2}$$

By Cauchy Residue theorem, $\int_C \frac{4}{(z-4)^3(z-2)}dz = 2\pi i\left(-\frac{1}{2}\right) = -\pi i$

17. Find the poles and residues of $f(z) = \frac{z}{z^2 - 3z + 2}$.

Solution: The poles of f (z) are obtained by

$$z^2 - 3z + 2 = 0 \Rightarrow (z-1)(z-2) = 0$$

$\therefore$ The poles are $z = 1, 2$ are simple poles

$$[\text{Res of f(z)}]_{z=1} = \underset{z\to1}{Lt}(z-1)\frac{z}{(z-1)(z-2)} = -1$$

$$[\text{Res of f(z)}]_{z=2} = \underset{z\to2}{Lt}(z-2)\frac{z}{(z-1)(z-2)} = 2$$

18. Calculate the residue of $f(z)=\frac{1-e^{2z}}{z^4}$ at the poles. (NOV / DEC 2014)

Solution: Given, $f(z)=\frac{1-e^{2z}}{z^4}$. Here $z=0$ is a pole of order 4

$$\therefore [\text{Res of } f(z)]_{z=0} = \underset{z\to0}{lt}\frac{1}{3!}\frac{d^3}{dz^3}\left[(z-0)^4\frac{1-e^{2z}}{z^4}\right]$$

$$= \underset{z\to0}{lt}\frac{1}{3!}\frac{d^3}{dz^3}\left[1-e^{2z}\right] = \underset{z\to0}{lt}\frac{1}{3!}\frac{d^2}{dz^2}\left[-2e^{2z}\right]$$

$$= \underset{z\to0}{lt}\frac{1}{6}\frac{d}{dz}\left[-4e^{2z}\right] = \frac{1}{6}\underset{z\to0}{lt}\left[-8e^{2z}\right] = \frac{-4}{3}$$

19. Express $\int_0^{\pi}\frac{d\theta}{2\cos\theta+\sin\theta}$ as complex integration. (DEC/JAN 2016)

Solution: Here $z=e^{i\theta}, dz=ie^{i\theta}d\theta \Rightarrow d\theta=\frac{1}{iz}dz$ $\quad \cos\theta=\frac{z^2+1}{2z}$ *and* $\sin\theta=\frac{z^2-1}{2iz}$

$$\frac{1}{2}\int_0^{2\pi}\frac{d\theta}{2\cos\theta+\sin\theta}=\frac{1}{2}\int_C\frac{\frac{dz}{iz}}{2\left(\frac{z^2+1}{2z}\right)+\left(\frac{z^2-1}{2iz}\right)}=\frac{1}{2}\int_C\frac{\frac{dz}{iz}}{\left(\frac{2iz^2+2i+z^2-1}{2iz}\right)}, \quad C:|z|=1$$

$$\Rightarrow \int_C\frac{dz}{2iz^2+2i+z^2-1}=\int_C\frac{dz}{z^2+2iz^2+(2i-1)}$$

20. Define Essential singularity with an example. (DEC/JAN 2016)

Solution: If the principal part of Laurents series contains an infinite number of non - zero terms, then $z = z_0$ is known as essential singularity.

Example: $f(z)=e^{\frac{1}{z}}=1+\frac{\frac{1}{z}}{1!}+\frac{\frac{1}{z^2}}{2!}+\ldots$ has z = 0 as an essential singularity. Since f(z) is an infinite series of negative powers of z.

PART – B

1. Evaluate $\int_C\frac{z+1}{(z-1)\ (z-3)}dz$, where C is | z | = 2 by Cauchy Integral Formula. (MAY / JUN 2016)

Solution :

$$\int_C\frac{z+1}{(z-1)\ (z-3)}dz=\int_C\frac{\frac{z+1}{z-3}}{(z-1)}dz$$

$$=2\pi i\, f(1)$$

Since z=1 lies inside the circle **| z | = 2**

z=3 lies outside the circle **| z | = 2**

Here $f(z)=\frac{z+1}{z-3}$

$f(1)=-1$

Hence by Cauchy integral formula

$$\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^n(a)$$

$$= 2\pi i \, f(1)$$

$$= -2\pi i$$

2. Evaluate $\int_c \frac{z^2}{(z-1)^2(z+2)} dz$ **where C is** $|z| = 3$**.** **(APR / MAY 2015)**

Solution :

Here $z = 1$ is a pole lies inside the circle

$z = -2$ is a pole lies inside the circle

$$\therefore \int_c \frac{z^2}{(z-1)^2(z+2)} dz$$

$$\operatorname{Res}_{z=1} f(z) = \frac{1}{(2-1)!} \lim_{z\to 1} \frac{d}{dz}\left[(z-1)^2 . f(z)\right]$$

$$= \lim_{z\to 1} \frac{d}{dz}\left[(z-1)^2 . \frac{z^2}{(z-1)^2(z+2)}\right]$$

$$= \lim_{z\to 1} \frac{d}{dz}\left[\frac{z^2}{(z+2)}\right] = \lim_{z\to 1}\left[\frac{z^2+4z}{(z+2)^2}\right]$$

$$= \frac{5}{9}$$

$$\operatorname{Res}_{z=-2} f(z) = \lim_{z\to 1}\left[(z+1). f(z)\right]$$

$$= \lim_{z\to 1}\left[(z+1). \frac{z^2}{(z-1)^2(z+2)}\right]$$

$$= \frac{4}{9}$$

By Cauchy Residue theorem,

$$\int_c \frac{z^2}{(z-1)^2(z+2)} dz = 2\pi i \,(sum\, of\ residues)$$

$$= 2\pi i\left(\frac{5}{9}+\frac{4}{9}\right) = 2\pi i$$

3. Evaluate $\int_C \frac{(z+1)\, dz}{(z-1)(z-2)^2}$, **where C is the circle** $| z - 2 | = \frac{1}{2}$ **by Cauchy Residue Theorem.**

(APR / MAY 2017)

Solution :

The poles are obtained by $(z - 1)(z - 2)^2 = 0$

$\Rightarrow$ z = 1 is a simple pole and z = 2 is a double pole.

C is the circle $|z-2| = \frac{1}{2}$

For z = 1, | z – 2 | = | 1 – 2 | = | –1 | = 1 > 1/2

For z = 2, | z – 2 | = | 2 – 2 | = 0 < ½

z = 1 lies outside C and z = 2 lies inside C.

$$\underset{z=2}{\text{Res}}\, f(z) = \underset{z\to 2}{lt}\frac{d}{dz}(z-2)^2\frac{z}{(z-1)(z-2)^2} = \underset{z\to 2}{lt}\frac{z-1-z}{(z-1)^2} = -1$$

By Cauchy Residue theorem,

$$\int_C \frac{z\,dz}{(z-1)(z-2)^2} = 2\pi i(-1) = -2\pi i$$

4. **Obtain Laurent's Series to represent the function** $\frac{z^2-1}{(z+2)(z+3)}$ **in the region** $|z|<2$ and $2<|z|<3$

Solution :

$$Let\ \ f(z) = \frac{z^2-1}{(z+2)(z+3)} = 1+\frac{3}{z+2}-\frac{8}{z+3}$$

(i) Given | z | < 2 $\Rightarrow \left|\frac{z}{2}\right| < 1\ and\ \left|\frac{z}{3}\right| < 1$

$$f(z) = 1+\frac{3}{2\left(1+\frac{z}{2}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1+\frac{3}{2}\left(1+\frac{z}{2}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1+\frac{3}{2}\left(1-\frac{z}{2}+\left(\frac{z}{2}\right)^2-\left(\frac{z}{2}\right)^3+\ldots\right)-\frac{8}{3}\left(1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\ldots\right)$$

(ii) Given 2 < | z | < 3

This region is annular about z = 0 and f (z) is analytic in this region.

| z | < 3 $\Rightarrow \left|\frac{z}{3}\right| < 1$ and 2 < | z | $\Rightarrow \left|\frac{2}{z}\right| < 1$

$$f(z) = 1+\frac{3}{z\left(1+\frac{2}{z}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$=1+\frac{3}{z}\left(1-\frac{2}{z}+\left(\frac{2}{z}\right)^{2}-\left(\frac{2}{z}\right)^{3}+\ldots\right)-\frac{8}{3}\left(1-\frac{z}{3}+\left(\frac{z}{3}\right)^{2}-\left(\frac{z}{3}\right)^{3}+\ldots\right)$$

$$=1+3\left[z^{-1}-2z^{-2}+4z^{-3}-8z^{-4}+\ldots\right]-\frac{8}{3}\left(1-\frac{z}{3}+\left(\frac{z}{3}\right)^{2}-\left(\frac{z}{3}\right)^{3}+\ldots\right)$$

5. Find the Laurent's series expansion of $\frac{1}{(z-2)(z-1)}$ valid in the regions $|z|>2$ and $0<|z-1|<1$

(APR / MAY 2017)

Solution :

(i). Let $f(z)=\frac{z}{(z+1)(z+2)}$

Now $\frac{z}{(z+1)(z+2)}=\frac{A}{z+1}+\frac{B}{z+2}$

$z=A(z+2)+B(z+1)$

Put $z=-1$

$A=-1$

Put $z=-2$

$B=1$

$\therefore f(z)=\frac{-1}{z+1}+\frac{2}{z+2}$

Given $|z|>2,\ 2<|z|$ i.e., $\left|\frac{2}{z}\right|<1\Rightarrow\frac{1}{|z|}<1$

$$\therefore f(z)=\frac{-1}{z+1}+\frac{2}{z+2}$$

$$=\frac{-1}{z\left(1+\frac{1}{z}\right)}+\frac{2}{z\left(1+\frac{2}{z}\right)}$$

$$=\frac{-1}{z}\left(1+\frac{1}{z}\right)^{-1}+\frac{2}{z}\left(1+\frac{2}{z}\right)^{-1}$$

(ii). $|z+1|<1$

Let $u=z+1$

i.e., $|u|<1$

$$f(z)=\frac{-1}{z+1}+\frac{2}{z+2}$$

$$=\frac{-1}{u}+\frac{2}{1+u}$$

$$=\frac{-1}{u}+2(1+u)^{-1}$$

$$=\frac{-1}{u}+2(1-u+u^{2}-\ldots)$$

$$=\frac{-1}{1+z}+2\left[1-(1+z)+(1+z)^{2}-\ldots\right]$$

6. Find the Laurent's series expansion of f(z) = $\frac{z^{2}-4z+2}{z^{3}-2z^{2}-5z+6}$ in 3 < | z + 2 | < 5 **(JAN 2016)**

Solution :

Solution:

The singular points are z = 1, z =32, z = –2

$$f(z)=\left(\frac{z^2-4z+2}{z^3-2z^2-5z+6}\right)$$

$$\frac{z^2-4z+2}{(z-1)(z+2)(z-3)}=\frac{A}{(z-1)}+\frac{B}{(z+2)}+\frac{C}{(z-3)}$$

$$A(z+2)(z-3)+B(z-1)(z-3)+C(z-1)(z+2)=z^2-4z+2$$

Put $z=1,\quad A(3)(-2)=1-4+2\Rightarrow -6A=-1\Rightarrow A=\frac{1}{6}$

Put $z=-2,\quad B(-3)(-5)=4+8+2\Rightarrow 15B=14\Rightarrow B=\frac{14}{15}$

Put $z=3,\quad C(2)(5)=9-12+2\Rightarrow 10C=-1\Rightarrow C=\frac{-1}{10}$

$$\frac{z^2-4z+2}{(z-1)(z+2)(z-3)}=\frac{1/6}{(z-1)}+\frac{14/15}{(z+2)}-\frac{2/10}{(z-3)}$$

$$3<|z+2|<5$$

Put $z+2=u\Rightarrow 3<|u|<5\Rightarrow \frac{3}{|u|}<1,\frac{|u|}{5}<1$

$$\frac{z^2-4z+2}{(z-1)(z+2)(z-3)}=\frac{1/6}{(z-1)}+\frac{14/15}{(z+2)}-\frac{2/10}{(z-3)}$$

$$=\frac{1/6}{(u-3)}+\frac{14/15}{(u)}-\frac{2/10}{(u-5)}$$

$$=\frac{1/6}{-3(1-u/3)}+\frac{14/15}{(u)}-\frac{2/10}{-5(1-u/5)}$$

$$=-\frac{1}{18}(1-u/3)^{-1}+\frac{14}{15u}+\frac{1}{50}(1-u/5)^{-1}$$

$$=-\frac{1}{18}\left(1+\frac{u}{3}+(\frac{u}{3})^2+\ldots\right)+\frac{14}{15u}+\frac{1}{50}\left(1+\frac{u}{5}+(\frac{u}{5})^2+\ldots\right)$$

$$f(z)=-\frac{1}{18}\left(1+\frac{z+2}{3}+(\frac{z+2}{3})^2+\ldots\right)+\frac{14}{15(z+2)}+\frac{1}{50}\left(1+\frac{z+2}{5}+(\frac{z+2}{5})^2+\ldots\right)$$

7. **Evaluate** $\int_0^{2\pi}\frac{d\theta}{13+12\cos\theta}$ **by using Contour integration.** **(MAY / JUN 2016)**

Solution :

Consider the unit circle | z | = 1 as contour C.

Put $z = e^{i\theta}$, *then* $\frac{1}{z} = e^{-i\theta}$

$$\therefore d\theta = \frac{dz}{iz},\ \sin\theta = \frac{z - \frac{1}{z}}{2i} = \frac{z^2 - 1}{2iz}$$

$$\therefore I = \int_C \frac{\frac{dz}{iz}}{13 + 5\frac{(z^2-1)}{2iz}} = \int_C \frac{\frac{dz}{iz}}{\frac{26iz + 5z^2 - 5}{2iz}} = 2\int_C \frac{dz}{5z^2 + 26iz - 5}$$

Let $f(z) = \frac{1}{5z^2 + 26iz - 5}$ $\qquad \therefore I = 2\int_C f(z)dz$

The poles of f(z) are given by $5z^2 + 26iz - 5 = 0$

$$z = \frac{-26i \pm \sqrt{(26i)^2 - 4.5(-5)}}{10} = \frac{-26i \pm \sqrt{-676 + 100}}{10} = \frac{-26i \pm \sqrt{-576}}{10} = \frac{-26i + 24i}{10} = -\frac{i}{5}, -5i$$

which are simple poles.

Now $5z^2 + 26iz - 5 = 5\left(z + \frac{i}{5}\right)(z + 5i)$

Since $\left|\frac{-i}{5}\right| = \frac{1}{5} < 1$, *the pole* $z = \frac{-i}{5}$ *lies inside C*

and $|-5i| = 5 > 1$, $\therefore$ *the pole* $z = -5i$ *lies outside C.*

$$\text{Now } R\left(-\frac{i}{5}\right) = \lim_{z \to \frac{-i}{5}} \left(z + \frac{i}{5}\right) f(z) = \lim_{z \to \frac{-i}{5}} \left(z + \frac{i}{5}\right) \frac{1}{5\left(z + \frac{i}{5}\right)(z + 5i)} = \lim_{z \to \frac{-i}{5}} \frac{1}{5(z + 5i)}$$

$$= \lim_{z \to \frac{-i}{5}} \frac{1}{5\left(-\frac{i}{5} + 5i\right)} = \frac{1}{24i}$$

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i\left(\frac{1}{24i}\right) = \frac{\pi}{12}$$

$$\therefore I = 2.\frac{\pi}{12} = \frac{\pi}{6}$$

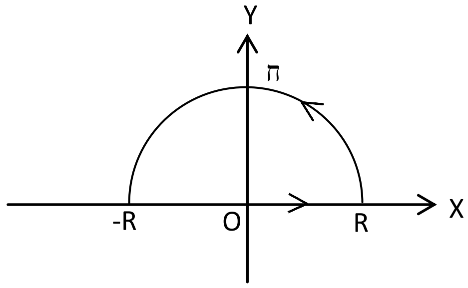
8. Evaluate $\int_0^\infty \frac{dx}{(x^2 + a^2)^2}, (a > 0)$ **using contour integration** **(APR / MAY 2017)**

Solution :

Let $\int_C \phi(z)dz = \int_C \frac{dz}{(z^2 + 1)^2}$

Where $\phi(z) = \frac{1}{(z^2 + 1)^2}$

Here C is the semicircle Γ bounded by the diameter $[-R, R]$



By Cauchy residue theorem,

$$\int_C \phi(z)\,dz = \int_{-R}^{R} \phi(x)\,dx + \int_\Gamma \phi(z)\,dz ......(1)$$

To evaluate of $\int_C \phi(z)\,dz$

The poles of $\phi(z) = \dfrac{1}{(z^2+1)^2}$ is the solution of $(z^2+1)^2 = 0$

i.e., $(z+i)^2 (z-i)^2 = 0$

i.e., the poles are $z = i,\ z = -i$

$z = i$ lies with inside the semi circle

$z = -i$ lies outside the semi circle

Now $[Res\,\phi(z)]_{z=i} = \underset{z \to i}{Lt} \dfrac{1}{1!} \dfrac{d}{dz} (z-i)^2 \phi(z)$

$$= \underset{z \to i}{Lt} \frac{1}{1!}\left[(z-i)^2 \frac{1}{(z^2+1)^2}\right] = \underset{z \to i}{Lt} \frac{1}{1!} \frac{d}{dz}\left[\frac{1}{(z+i)^2}\right] \qquad \because z^2+1 = (z+i)(z-i)$$

$$= \underset{z \to i}{Lt} \frac{-2}{(z+i)^3}$$

$$= \frac{-2}{i+i} = \frac{-2}{(2i)^3} = \frac{1}{4i}$$

$$\therefore \int_C \phi(z)\,dz = 2\pi i\left[Sum\ of\ residues\ of\ \phi(z) \text{ at } its\ poles\ which\ lies\ in\, C\right]$$

$$= 2\pi i\left[\frac{1}{4i}\right] = \frac{\pi}{2} ....(2)$$

Let $R \to \infty$, then $|z| \to \infty$ so that $\phi(z) = 0$

$$\therefore \underset{|z| \to \infty}{Lt} \int_\Gamma \phi(z)\,dz = 0 ....(3)$$

Sub (2) and (3) in (1)

$$\int_C \phi(z)\,dz = \int_{-\infty}^{\infty} \phi(x)\,dx$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2}$$

$$\Rightarrow 2\int_0^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2}$$

$$\Rightarrow \int_0^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}.$$

9. Evaluate $\int_0^{2\pi} \frac{d\theta}{1-2x\sin\theta+x^2}$, | x | < 1 **(APR / MAY 2017)**

Solution :

Let $z = e^{i\theta}$, $dz = i e^{i\theta} \Rightarrow d\theta = \frac{dz}{iz}$, $\sin\theta = \frac{z^2-1}{2iz}$

$$\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2} = \int_C \frac{\left(\frac{dz}{iz}\right)}{1-2p\left(\frac{z^2-1}{2iz}\right)+a^2} \quad ,\text{C is } |z|=1$$

$$= \int_C \frac{dz}{iz - p(z^2-1)+iza^2} = -\int_C \frac{dz}{pz^2 - iz(p^2+1)-p} = -\frac{1}{p}\int_C \frac{dz}{z^2 - iz\left(p+\frac{1}{p}\right)-1}$$

$$\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2} = -\frac{1}{p}\int_C \frac{dz}{(z-ip)\left(z-\frac{i}{p}\right)} \quad ....(1)$$

The poles are given by $z = ip$ & $z = \frac{i}{p}$

| z | = | i p | = p < 1. $\therefore$ z= ip lies inside C and $z = \frac{i}{p}$ lies outside C.

$$\therefore [\text{Res of } f(z)]_{z=ip} = \underset{z\to ip}{\text{Lt}} (z-ip)\left[\frac{1}{(z-ip)\left(z-\frac{i}{p}\right)}\right]$$

$$= \underset{z\to ip}{\text{Lt}} \left(\frac{1}{z-\frac{i}{p}}\right) = \frac{1}{i\left(p-\frac{1}{p}\right)} = \frac{ip}{1-p^2}$$

By Cauchy Residue Theorem $\int_C \frac{dz}{(z-ip)\left(z-\frac{i}{p}\right)} = 2\pi i\left(\frac{ip}{1-p^2}\right) = \frac{-2\pi p}{1-p^2}$

From (1) $\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2} = -\frac{1}{p}\left(-\frac{2\pi p}{1-p^2}\right) = \frac{2\pi}{1-p^2}$

10. Evaluate $\int_0^{\infty} \frac{x\sin x \; dx}{(x^2+a^2)}$, where a > 0 **(MAY / JUN 2016)**

Solution:

$$2\int_0^{\infty} \frac{x\sin x}{x^2+a^2}dx = \int_{-\infty}^{\infty} \frac{x\sin x}{x^2+a^2}dx$$

$$\int_0^{\infty} \frac{x\sin x}{x^2+a^2}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+a^2}dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{z\sin z}{z^2+a^2}dz$$

$$=\frac{1}{2}I........(1)$$

Now $z\sin z$ is the imaginary part of ze^{iz}

$$\therefore \mathrm{I}=\int_{-\infty}^{\infty}\frac{z\sin z}{z^2+a^2}dz$$

$$=\mathrm{I.P.}\int_{-\infty}^{\infty}\frac{ze^{iz}}{z^2+a^2}dz$$

Let $\phi(z)=\frac{z\,e^{iz}}{z^2+a^2}=\frac{ze^{iz}}{(z+ia)(z-ia)}$

The poles are $z=-ia$, $z=ia$
Now the poles $z=ia$ lies in the upper half – plane
But $z=-ia$ lies in the lower half – plane.
Hence

$$\left[Res\phi(z)\right]_{z=ia}=\underset{z\to ia}{Lt}(z-ia)\frac{ze^{iz}}{(z+ia)(z-ia)}$$

$$=\underset{z\to ia}{Lt}\frac{ze^{iz}}{(z+ia)}$$

$$=\frac{iae^{-a}}{2ia}=\frac{e^{-a}}{2}$$

$$\therefore\int_{-\infty}^{\infty}\frac{ze^{iz}}{z^2+a^2}dz=2\pi i\left[\text{Sum of the residues at each poles in the upper half plane}\right]$$

$$=2\pi i\left[\frac{e^{-a}}{2}\right]$$

$$=\pi ie^{-a}$$

$$\mathrm{I=I.P.\ of}\int_{-\infty}^{\infty}\frac{ze^{iz}}{z^2+a^2}dz$$

$$=\mathrm{I.P.\ of}\left(\pi ie^{-a}\right)$$

$$\mathrm{I}=\pi e^{-a}...........(2)$$

Sub (2) in (1)

$$\int_{0}^{\infty}\frac{x\sin x}{x^2+a^2}dx=\frac{1}{2}x=\frac{1}{2}\pi e^{-a}$$

UNIT- V LAPLACE TRANFORM

PART A

1. **State the sufficient condition for the existence of Laplace transforms.** **(APR / MAY 2015, 2017)**

   Solution: The Laplace transform of $f(t)$ exists if

   (i) $f(t)$ is piecewise continuous in $[a, b]$ where $a > 0$.

   (ii) $f(t)$ is of exponential order.

2. **Find the Laplace transform of $e^{-2t}t^{1/2}$.**

   Solution: $L\left[e^{-2t}t^{1/2}\right]=L\left[t^{1/2}\right]_{s\to s+2}$

$$\because \text{If } L[f(t)]=F(s), \text{then } L\left[e^{-at} f(t)\right]=F(s)\big|_{s\to s+a}$$

$$=\left[\frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{3/2}}\right]_{s\to s+2} \quad =\left[\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{3/2}}\right]_{s\to s+2}$$

$$=\frac{\frac{1}{2}\sqrt{\pi}}{(s+2)^{3/2}} \qquad \left(\because \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}, \Gamma n+1=n\Gamma n\right)$$

3. If L[$f(t)$] = F(s), Prove that $L\left[f\left(\frac{t}{5}\right)\right]=5F(5s)$.

Solution: $L[f(t)]=\int_0^\infty e^{-st} f(t)\,dt$

$$L\left[f\left(\frac{t}{5}\right)\right]=\int_0^\infty e^{-st} f\left(\frac{t}{5}\right)dt$$

put $\frac{t}{5}=u \Rightarrow$ 5du = dt

$$\therefore L\left[f\left(\frac{t}{5}\right)\right]=\int_0^\infty e^{-5su} f(u)5du$$

$$=5\int_0^\infty e^{-(5s)u} f(u)\,du$$

= 5 F(5s)

4. Find the Laplace transform of unit step function.

Solution: The Unit step function is $u_a(t)=\begin{cases}0, & t<a\\ 1, & t>a,\ a\ge 0\end{cases}$

The Laplace transform $L[f(t)]=\int_0^\infty e^{-st} f(t)\,dt=\int_a^\infty e^{-st}(1)\,dt=\left[\frac{e^{-st}}{-s}\right]_a^\infty=-\frac{1}{s}\left[e^{-\infty}-e^{-as}\right]=\frac{e^{-as}}{s}$.

5. Prove that $L\left(\int_0^t f(t)dt\right)=\frac{F(s)}{s}$**, where** $L[f(t)]=F(s)$ **(DEC-2016)**

Solution: Let $F(t)=\int_0^t f(t)\,dt$

$$\therefore F'(t)=f(t)$$

$$\therefore L[F'(t)]=sL[F(t)]-F(0)=sL[F(t)]-0$$

$$L[f(t)]=sL[F(t)]=sL\left[\int_0^t f(t)\,dt\right]$$

$$\therefore L\left(\int_0^t f(t)dt\right)=\frac{F(s)}{s}$$

6. Does $L\left[\frac{\cos at}{t}\right]$ **exist?**

Solution: $\underset{t\to 0}{Lt}\dfrac{f(t)}{t} = \underset{t\to 0}{Lt}\dfrac{\cos at}{t} = \dfrac{1}{0} = \infty$

$\therefore \quad L\left[\dfrac{\cos at}{t}\right]$ does not exist.

7. Obtain the Laplace transform of $\sin 2t - 2t\cos 2t$.

Solution: $L[\sin 2t - 2t\cos 2t] = L[\sin 2t] - 2L[t\cos 2t] = L[\sin 2t] - 2\left(-\dfrac{d}{ds}L[\cos 2t]\right)$

$$= \frac{2}{s^2+4} + 2\frac{d}{ds}\left(\frac{s}{s^2+4}\right) = \frac{2}{s^2+4} + 2\left(\frac{(s^2+4)(1) - s(2s)}{(s^2+4)^2}\right)$$

$$= \frac{2(s^2+4) + 2(4-s^2)}{(s^2+4)^2}$$

$$\therefore \quad L[\sin 2t - 2t\cos 2t] = \frac{16}{(s^2+4)^2}$$

8. Find $L^{-1}\left[\dfrac{s+2}{s^2+2s+2}\right]$

Solution: $L^{-1}\left[\dfrac{s+2}{s^2+2s+2}\right] = L^{-1}\left[\dfrac{(s+1)+1}{(s+1)^2+1}\right]$ $\qquad \because L^{-1}[F(s+a)] = e^{-at}L^{-1}[F(s)]$

$$= L^{-1}\left[\frac{(s+1)}{(s+1)^2+1}\right] + L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= e^{-t}\left(L^{-1}\left[\frac{s}{s^2+1}\right] + L^{-1}\left[\frac{1}{s^2+1}\right]\right)$$

$$\therefore \quad L^{-1}\left[\frac{s+2}{s^2+2s+2}\right] = e^{-t}(\cos t + \sin t)$$

9. What is the Laplace transform of $f(t), 0 < t < 10$ with $f(t) = f(t+10)$?

Solution: Given $f(t)$ is a periodic function with period p

$$L[f(t)] = \frac{1}{1-e^{-ps}}\int_0^p e^{-st} f(t)dt$$

Put $p = 10$, $L[f(t)] = \dfrac{1}{1-e^{-10s}}\displaystyle\int_0^{10} e^{-st} f(t)dt$

10. Using Laplace transform, Evaluate $\displaystyle\int_0^\infty t\, e^{-2t}\sin t\, dt$ (APR / MAY 2015)

Solution: $\displaystyle\int_0^\infty e^{-2t} f(t)dt = \left[\int_0^\infty e^{-st} f(t)dt\right]_{s=2} = [L[t\sin t]]_{s=2} = \left[-\frac{d}{ds}L[\sin t]\right]_{s=2}$

$$= -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = -\left(\frac{-2s}{(s^2+1)^2}\right) = \frac{4}{25}$$

11. Find $L^{-1}\left(\dfrac{s}{s^2+4s+5}\right)$ (MAY-JUNE 2016)

Solution:

$$L^{-1}\left(\frac{s}{s^2+4s+5}\right)=L^{-1}\left(\frac{(s+2)-2}{(s+2)^2+1}\right)=e^{-2t}L^{-1}\left(\frac{s-2}{s^2+1}\right)$$
$$=e^{-2t}\left[L^{-1}\left(\frac{s}{s^2+1}\right)-2L^{-1}\left(\frac{1}{s^2+1}\right)\right]$$
$$=e^{-2t}\left[\cos t-2\sin t\right]$$

12. Find the Laplace transform $sin^3(2t)$

Solution: $L\left[sin^3(2t)\right]=\frac{1}{4}L[3\sin 2t-\sin 6t]=\frac{3}{4}L[\sin 2t]-\frac{1}{4}L[\sin 6t]$ $\left(\because \sin^3 t=\frac{1}{4}[3\sin t-\sin 3t]\right)$

$$=\frac{3}{4}\left(\frac{2}{s^2+4}\right)-\frac{1}{4}\left(\frac{6}{s^2+36}\right)$$
$$=\frac{6}{4}\left(\frac{1}{s^2+4}-\frac{1}{s^2+36}\right).$$

13. Find $L^{-1}\left[\tan^{-1}\left(\frac{1}{s}\right)\right]$

Solution: Let $F(s)=\tan^{-1}\left(\frac{1}{s}\right)$

$$F'(s)=\frac{1}{1+(1/s)^2}\left(\frac{-1}{s^2}\right)=\frac{-1}{s^2+1}$$

By property $L^{-1}\left[F'(s)\right]=-L^{-1}\left[\frac{1}{s^2+1}\right]=-\sin t$

$$\therefore L^{-1}(F'(s))=-\sin t;\qquad L^{-1}(F(s))=\frac{-1}{t}L^{-1}\left[F'(s)\right]$$
$$\therefore L^{-1}\left[\tan^{-1}\left(\frac{1}{s}\right)\right]=\frac{\sin t}{t}$$

14. Solve using Laplace transform $\frac{dy}{dt}+y=e^{-t}$ given that y(0) = 0.

Solution: Taking L.T. on both sides, we get $L\left[y'(t)\right]+L\left[y(t)\right]=L\left[e^{-t}\right]$

$$sL\left[y(t)\right]-y(0)+L\left[y(t)\right]=L\left[e^{-t}\right]$$
$$sL\left[y(t)\right]-0+L\left[y(t)\right]=\frac{1}{s+1}$$
$$(s+1)\,L[y(t)]=\frac{1}{s+1}$$
$$L[y(t)]=\frac{1}{(s+1)^2}$$
$$\therefore y(t)=L^{-1}\left(\frac{1}{(s+1)^2}\right)=e^{-t}L\left(\frac{1}{s^2}\right)=e^{-t}\,t \qquad \left(\because L\left[e^{-at}f(t)\right]=F(s+a)\right)$$

15. Give an example for a function that do not have Laplace transform.

Solution: Consider $f(t)=e^{t^2}$, since $\underset{t\to\infty}{Lt}\ e^{-st}e^{t^2}=\infty$, hence e^{t^2} is not exponential order.

Hence $f(t) = e^{t^2}$ does not have Laplace transform.

16. Can F(s) = $\frac{s^3}{(s+1)^2}$ be the Laplace transform of some *f*(*t*)?

Solution: $\underset{s\to\infty}{Lt} F(s) = \underset{s\to\infty}{Lt} \frac{s^3}{(s+1)^2} \neq 0$

Hence *F*(*s*) cannot be Laplace transform of *f*(*t*).

17. Evaluate $\int_0^t \sin u \cos(t-u)\,du$ using Laplace Transform.

Solution: Let $L\left[\int_0^t \sin u \cos(t-u)\,du\right] = L[\sin t * \cos t]$

$= L[\sin t]\ \ L[\cos t]$ (by Convolution theorem)

$= \frac{1}{(s^2+1)}\frac{s}{(s^2+1)} = \frac{s}{(s^2+1)^2}$

$$\int_0^t \sin u \cos(t-u)\,du = L^{-1}\left[\frac{s}{(s^2+1)^2}\right] = \frac{1}{2}L^{-1}\left[\frac{2s}{(s^2+1)^2}\right] = \frac{t}{2}\sin t \quad \left(\because L^{-1}\left[\frac{2s}{(s^2+a^2)^2}\right] = t\sin at\right)$$

18. Give an example for a function having Laplace transform but not satisfying the continuity condition.

Solution: $f(t) = t^{\frac{-1}{2}}$ has Laplace transform even though it does not satisfy the continuity condition. (i.e.) It is not piecewise continuous in (0,∞) as $\underset{t\to 0}{Lt} f(t) = \infty$

19. Define a Periodic function with example.

Definition: A function *f* (*t*) is said to be periodic function if *f*(*t* + *p*) = *f*(*t*) for all *t*. The least value of *p* > 0 is called the period of *f*(*t*). For example, sin *t* and cos *t* are periodic functions with period 2π .

20. State Convolution theorem on Laplace Transform. **(MAY-JUNE 2017)**

Statement: The Laplace transform of convolution of two functions is equal to the product of their Laplace transforms.

(i.e) $L[f(t)*g(t)] = L[f(t)]\ L[g(t)]$.

PART B

1. (a) (i) Find the Laplace transform of $f(t) = te^{-2t}\cos 3t$ **(APR/MAY 2017)**

(ii) Find $L^{-1}\left\{\log\left(\frac{s^2+4}{(s-2)^2}\right)\right\}$

Solution:

(a) (i) $L[tf(t)] = -\frac{d}{ds}F(s)$

$$L[te^{-2t}\cos 3t] = -\frac{d}{ds}L[e^{-2t}\cos 3t] = -\frac{d}{ds}\left[[L[\cos 3t]]_{s\to s+2}\right]$$

$$= -\frac{d}{ds}\left[\left[\frac{s}{s^2+9}\right]_{s\to s+2}\right] = -\frac{d}{ds}\left[\frac{s+2}{(s+2)^2+9}\right]$$

$$= -\left[\frac{((s+2)^2+9)(1)-(s+2)2(s+2)}{((s+2)^2+9)^2}\right] = \left[\frac{((s+2)^2-9)}{((s+2)^2+9)^2}\right]$$

(ii) $L^{-1}\left\{\log\left(\frac{s^2+4}{(s-2)^2}\right)\right\} = f(t)$

$$F(s)=\log\left(\frac{s^2+4}{(s-2)^2}\right) = \log(s^2+4)-\log(s-2)^2$$

$$L[tf(t)]=-\frac{d}{ds}F(s) \Rightarrow L[tf(t)]=-\frac{d}{ds}\left(\log(s^2+4)-2\log(s-2)\right)$$

$$=-\frac{2s}{s^2+4}+\frac{2}{(s-2)} \Rightarrow tf(t)=L^{-1}\left[\frac{-2s}{s^2+4}+\frac{2}{(s-2)}\right] \Rightarrow f(t)=\frac{2}{t}\left(e^{2t}-\cos 2t\right)$$

(b) Using Laplace transform, solve the differential equation $\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=e^{-t}, y(0)=1, y'(0)=0$

Solution :

$$\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=e^{-t}, y(0)=1, y'(0)=0$$

Taking Laplace transforms on both sides, we have

$$L\left[\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y\right]=L[e^{-t}]$$

$$s^2L(y(t))-sy(0)-y'(0)+3\left(sL(y(t))-y(0)\right)+2L(y(t)=\frac{1}{s+1}$$

$$s^2L(y(t))-s-0+\left(3sL(y(t))-3\right)+2L(y(t)=\frac{1}{s+1}$$

$$L(y(t))(s^2+3s+2)=\frac{1}{s+1}+s+3$$

$$L(y(t))=\frac{s^2+4s+4}{(s+1)^2(s+2)}$$

$$y(t)=L^{-1}\left(\frac{s^2+4s+4}{(s+1)^2(s+2)}\right)$$

By partial fraction method:

$$\left(\frac{s^2+4s+4}{(s+1)^2(s+2)}\right)=\frac{A}{(s+1)}+\frac{B}{(s+1)^2}+\frac{C}{s+2}$$

By solving we get A=1, B=1, C=0

$$\Rightarrow L^{-1}\left(\frac{s^2+4s+4}{(s+1)^2(s+2)}\right)= L^{-1}\left(\frac{1}{(s+1)}+\frac{1}{(s+1)^2}\right)=e^{-t}+te^{-t}.$$

(or)

$$y(t)=L^{-1}\left(\frac{s^2+4s+4}{(s+1)^2(s+2)}\right)=L^{-1}\left(\frac{(s+2)^2}{(s+1)^2(s+2)}\right)=L^{-1}\left(\frac{(s+2)}{(s+1)^2}\right)=L^{-1}\left(\frac{(s+1+1)}{(s+1)^2}\right)=L^{-1}\left(\frac{1}{(s+1)}\right)+L^{-1}\left(\frac{1}{(s+1)^2}\right)$$

$$= e^{-t}+te^{-t}$$

2. **(a) Evaluate** $\int_0^\infty e^{-t}\left(\frac{\cos 2t - \cos 3t}{t}\right)dt$

Solution :

$$\int_0^\infty e^{-t}\left(\frac{\cos 2t-\cos 3t}{t}\right)dt = L\left(\frac{\cos 2t-\cos 3t}{t}\right)\Bigg|_{s=1}$$

$$= \int_s^\infty L(\cos 2t - \cos 3t)ds\Big|_{s=1} = \int_s^\infty\left(\frac{s}{s^2+4}-\frac{s}{s^2+9}\right)ds\Bigg|_{s=1} = \frac{1}{2}\log\left(\frac{10}{5}\right) = \log\sqrt{2}$$

(b) Apply convolution theorem to evaluate $L^{-1}\left\{\frac{s^2}{(s^2+4)(s^2+9)}\right\}$ **(APR/MAY 2017)**

Solution :

(a) $L^{-1}\left[\frac{s^2}{(s^2+4)(s^2+9)}\right] = L^{-1}\left[\frac{s}{(s^2+4)}\right] * L^{-1}\left[\frac{s}{(s^2+9)}\right]$

$$=\cos 2t * \cos 3t = \int_0^t \cos 2u \cos 3(t-u)du$$

$$= \frac{1}{2}\int_0^t[\cos(2u+3t-3u) + \cos(2u-3t+3u)]\,du$$

$$=\frac{1}{2}\int_0^t[\cos(3t-u)+\cos(5u-3t)]du$$

$$=\frac{1}{2}\left[\frac{\sin(3t-u)}{-1}+\frac{\sin(5u-3t)}{5}\right]_0^t = \frac{1}{2}\left(\left[\frac{\sin 2t}{-1}+\frac{\sin 2t}{5}\right]-\left[\frac{\sin 3t}{-1}-\frac{\sin 3t}{5}\right]\right)$$

$$=-\frac{4}{10}\sin 2t+\frac{6}{10}\sin 3t$$

$$=\frac{3}{5}\sin 3t-\frac{2}{5}\sin 2t$$

3. **(a) Verify initial and final value theorems for the function** $f(t) = 1 + e^{-t}(\sin t + \cos t)$

Solution :

(a) Initial value theorem states $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$$F(s) = L(1+e^{-t}(\sin t+\cos t)) = \frac{1}{s}+\frac{1}{(s+1)^2+1}+\frac{s+1}{(s+1)^2+1} = \frac{1}{s}+\frac{s+2}{(s+1)^2+1}$$

L.H.S: $\lim_{t\to 0} f(t) = \lim_{t\to 0}(1+e^{-t}(\sin t+\cos t)) = 2$

R.H.S: $\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\left(\frac{1}{s}+\frac{s+2}{(s+1)^2+1}\right) = \lim_{s\to\infty}\left(1+\frac{s(s+2)}{(s+1)^2+1}\right)$

$$\lim_{s\to\infty}\left(1+\frac{s^2\left(1+\frac{2}{s}\right)}{s^2\left(1+\frac{2}{s}+\frac{2}{s^2}\right)}\right)=2.$$

Final Value theorem states $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

L.H.S: $\lim_{t\to\infty} f(t) = \lim_{t\to\infty}(1+e^{-t}(\sin t+\cos t)) = 1$

R.H.S: $\lim_{s\to 0} sF(s) = = \lim_{s\to 0}\left(1+\frac{s(s+2)}{(s+1)^2+1}\right) = 1$

Hence Initial and Final Value theorems are verified.

(b) Find $L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right]$

Solution :

$$L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right]$$

By the method of partial fraction method:

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{A}{(s-2)} + \frac{B}{(s-2)^2} + \frac{C}{(s-2)^3} + \frac{D}{(s+1)}$$

$$5s^2-15s-11 = A(s-2)^3 + B(s+1)(s-2)^2 + C(s+1)(s-2) + D(s+1)$$

Equating the co- efficient on both sides we have

A=1/3, B=4, C = -7, D = -1/3

$$L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right] = L^{-1}\left[\frac{1/3}{(s-2)} + \frac{4}{(s-2)^2} + \frac{-7}{(s-2)^3} + \frac{-1/3}{(s+1)}\right]$$

$$= \frac{1}{3}e^{2t} + 4e^{2t}t - \frac{7}{2}e^{2t}t^2 - \frac{1}{3}e^{-t}$$

4. (a) Find the Laplace transform of f(t) = $\begin{cases} E & if\ 0<t<\frac{a}{2} \\ -E & if\ \frac{a}{2}<t<a \end{cases}$ **where f(t+a)=f(t) for all t**

Solution :

$$L[f(t)] = \frac{1}{1-e^{-ps}}\int_0^p e^{-st} f(t)dt$$

$$L[f(t)] = \frac{1}{1-e^{-as}}\int_0^a e^{-st} f(t)dt = \frac{1}{1-e^{-as}}\left[\int_0^{a/2} e^{-st}Edt - \int_{a/2}^a e^{-st}Edt\right]$$

$$= \frac{E}{1-e^{-as}}\left[\int_0^{a/2} e^{-st}dt - \int_{a/2}^a e^{-st}dt\right] = \frac{E}{1-e^{-as}}\left\{\left[\frac{e^{-st}}{-s}\right]_0^{a/2} - \left[\frac{e^{-st}}{-s}\right]_{a/2}^a\right\}$$

$$= \frac{E}{1-e^{-as}}\left\{\left[\frac{e^{-\frac{as}{2}}}{-s} + \frac{1}{s}\right] - \left[\frac{e^{-as}}{-s} + \frac{e^{\frac{-as}{2}}}{s}\right]\right\} = \frac{E}{1-e^{-as}}\left[\frac{1+e^{-as}-2e^{-as/2}}{s}\right]$$

$$= \frac{E}{\left(1-e^{\frac{-as}{2}}\right)\left(1+e^{\frac{-as}{2}}\right)}\left[\frac{\left(1-e^{-as/2}\right)^2}{s}\right] = \frac{E}{s}\left[\frac{\left(1-e^{-as/2}\right)}{\left(1+e^{\frac{-as}{2}}\right)}\right]$$

$$= \frac{E}{s}\frac{\left(1-e^{-as/2}\right)}{\left(1+e^{\frac{-as}{2}}\right)} = \frac{E}{s}\frac{\left(e^{as/4}-e^{-as/4}\right)}{\left(e^{as/4}+e^{\frac{-as}{4}}\right)} = \frac{E}{s}\tanh\left(\frac{as}{4}\right)$$

(b) Find the Laplace transform of $e^{-4t}\int_0^t t\ \sin 3t dt$

Solution :

$$L\left[e^{-4t}\int_0^t t\ \sin 3t dt\right] = L\left[\int_0^t t\sin 3t dt\right]\Bigg|_{s\to s+4}$$

$$= \left[\frac{1}{s}L(t\sin 3t)\right]\Bigg|_{s\to s+4} = \left[\frac{1}{s}\left(-\frac{d}{ds}L(\sin 3t)\right)\right]\Bigg|_{s\to s+4} = \left[\frac{1}{s}\left(-\frac{d}{ds}\left(\frac{3}{s^2+9}\right)\right)\right]\Bigg|_{s\to s+4}$$

$$= \left[\frac{1}{s}\left(\frac{6s}{(s^2+9)^2}\right)\right]\Bigg|_{s\to s+4} = \frac{1}{(s+4)}\left(\frac{6(s+4)}{\left((s+4)^2+9\right)^2}\right) = \left(\frac{6}{\left((s+4)^2+9\right)^2}\right)$$

5. (a) Find the Laplace transform of $\frac{1-\cos t}{t^2}$

Solution :

$$L\left(\frac{1-\cos t}{t^2}\right) = \int_s^\infty\int_s^\infty\left(\frac{1}{s}-\frac{s}{s^2+1}\right)ds\,ds$$

$$= \int_s^\infty\left[\log(s)-\frac{1}{2}\log(s^2+1)\right]_s^\infty ds$$

$$= \int_s^\infty\left[\frac{1}{2}\log(s^2)-\frac{1}{2}\log(s^2+1)\right]_s^\infty ds$$

$$= \frac{1}{2}\int_s^\infty\log\left(\frac{s^2}{s^2+1}\right)_s^\infty ds = \frac{1}{2}\int_s^\infty\log\left(\frac{s^2+1}{s^2}\right)ds$$

$$= \cot^{-1}s-\frac{s}{2}\log\left(\frac{s^2+1}{s}\right)$$

(b) Using Laplace transforms technique solve $y''+y'=t^2+2t$, **given** $y=4, y'=-2$ **when** $t=0$ **(MAY/JUNE 2016)**

Solution :

$y''+y'=t^2+2t$, **given** $y=4, y'=-2$ **when** $t=0$

Taking Laplace transform on both sides

$$L(y''+y')=L(t^2+2t)$$

$$s^2L(y(t))-sy(0)-y'(0)+\left(sL(y(t))-y(0)\right)=\frac{2}{s^3}+\frac{2}{s^2}$$

$$s^2L(y(t)) - 4s + 2 + \left(sL(y(t)) - 4\right) = \frac{2}{s^3} + \frac{2}{s^2}$$

$$(s^2 + s)L(y(t)) - 4s - 2 = \frac{2}{s^3} + \frac{2}{s^2}$$

$$(s^2 + s)L(y(t)) = \frac{2}{s^3} + \frac{2}{s^2} + 4s + 2$$

$$s(s+1)L(y(t)) = \frac{2}{s^3} + \frac{2}{s^2} + 4s + 2$$

$$L(y(t)) = \frac{1}{s(s+1)}\left[\frac{2}{s^3} + \frac{2}{s^2} + 4s + 2\right] = \frac{4s^4 + 2s^3 + 2s + 2}{s^4(s+1)}$$

$$y(t) = L^{-1}\left[\frac{2}{s} + \frac{2}{s^4} + \frac{2}{(s+1)}\right] = 2 + \frac{t^3}{3} + 2e^{-t}$$