# 1.Partial Differential Equation

An equation involving partial derivatives of two or more independent variables is called a Partial differential equation(PDE)

## Formation of PDE

PDE are formed in two ways

- by eliminating arbitrary constants.
- by eliminating arbitrary functions.

Notations:

If $z = f(x, y)$ then

$$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$$

$$r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$$

## Formation of PDE by elimination of arbitrary constants and arbitrary functions

1. If no.of arbitrary constants $\leq$ no.of independent variable then use p,q only.
2. If no.of arbitrary constants $>$ no.of independent variable then use p,q,r,s and t.

Form a PDE by eliminating arbitrary constants

1. $z = a(x + y) + b$

   Solution:

   Given $z = a(x + y) + b$. .........(1)    no.of a.c=no.of I.V

   Differentiate (1) par. w.r.t x and y,

   $p = \frac{\partial z}{\partial x} = a$. .....(2),    $q = \frac{\partial z}{\partial y} = a$. ......(3)

   From (2) & (3)$\Rightarrow$ $p = q$ which is the required p.d.e.

2. $z = a^2 x + ay^2 + b$

   Solution:

   Given $z = a^2 x + ay^2 + b$. .........(1)    no.of a.c=no.of I.V

Differentiate (1) par. w.r.t x and y,

$$p = \frac{\partial z}{\partial x} = a^2. .....(2)$$

$$\begin{aligned} q &= \frac{\partial z}{\partial y} = 2ay \\ \Rightarrow y &= \frac{q}{2a} \\ y^2 &= \frac{q^2}{4a^2} \\ y^2 &= \frac{q^2}{4p} \qquad \text{using (2)} \end{aligned}$$

$$\Rightarrow 4py^2 = q^2 \text{ which is the required p.d.e.}$$

3. $z = ax^n + by^n$

   Solution:

   Given $z = ax^n + by^n$. .........(1) no.of a.c=no.of I.V

   Differentiate (1) par. w.r.t x and y,

   $$p = \frac{\partial z}{\partial x} = nax^{n-1}, \qquad q = \frac{\partial z}{\partial y} = nby^{n-1}$$

   $$a = \frac{p}{nx^{n-1}}, \qquad b = \frac{q}{ny^{n-1}}$$

   Sub in (1)

   $$z = \frac{p}{nx^{n-1}}x^n + \frac{q}{ny^{n-1}}y^n$$

   $\Rightarrow nz = px + qy$ which is the required p.d.e.

4. $z = ax^2 + by^2$ Ans: $2z = px + qy$

5. $z = ax^3 + by^3$ Ans: $3z = px + qy$

6. $z = (x^2 + a)(y^2 + b)$

   Solution:

   Given $z = (x^2 + a)(y^2 + b)$. .........(1) no.of a.c=no.of I.V

   Differentiate (1) par. w.r.t x and y,

   $$p = \frac{\partial z}{\partial x} = 2x(y^2 + b), \qquad q = \frac{\partial z}{\partial y} = 2y(x^2 + a)$$

   $$y^2 + b = \frac{p}{2x}, \qquad x^2 + a = \frac{q}{2y}$$

   Sub in (1)

   $$z = \frac{q}{2y}\frac{p}{2x}$$

   $\Rightarrow 4xyz = pq$ which is the required p.d.e.

7. $z = (x^2 + a^2)(y^2 + b^2)$ Ans: $4xyz = pq$

8. $(x - a)^2 + (y - b)^2 + z^2 = r^2$

   Solution:

   Given $(x - a)^2 + (y - b)^2 + z^2 = r^2$. .....(1) no.of a.c=no.of I.V

   Differentiate (1) par. w.r.t x and y,

   $$2(x - a) + 2z\frac{\partial z}{\partial x} = 0, \qquad 2(y - b) + 2z\frac{\partial z}{\partial y} = 0$$

   $$x - a = -zp, \qquad y - b = -zq$$

Sub in (1)

$$(-zp)^2 + (-zq)^2 + z^2 = r^2$$
$$z^2p^2 + z^2q^2 + z^2 = r^2$$
$$\Rightarrow z^2(p^2 + q^2 + 1) = r^2 \text{ which is the required p.d.e.}$$

9. $(x-a)^2 + (y-b)^2 + z^2 = 1$ — Ans: $z^2(p^2 + q^2 + 1) = 1$

10. $z = (x-a)^2(y-b)^2$ — Ans: $4xyz = pq$

11. $z = (x-a)^2 + (y-b)^2$ — Ans:$p^2 + q^2 = 4z$

12. $z = (x+a)^3 + (y-b)^3$

    Solution:

    Given $z = (x+a)^3 + (y-b)^3$. .........(1) — no.of a.c=no.of I.V

    Differentiate (1) par. w.r.t x and y,

$$p = \frac{\partial z}{\partial x} = 3(x+a)^2, \qquad q = \frac{\partial z}{\partial y} = 3(y-b)^2$$
$$(x+a)^2 = \frac{p}{3}, \qquad (y-b)^2 = \frac{q}{3}$$
$$x + a = \left(\frac{p}{3}\right)^{\frac{1}{2}}, \qquad y - b = \left(\frac{q}{3}\right)^{\frac{1}{2}}$$
$$(x+a)^3 = \left(\frac{p}{3}\right)^{\frac{3}{2}}, \qquad (y-b)^3 = \left(\frac{q}{3}\right)^{\frac{3}{2}}$$

    Sub in (1)

$$\Rightarrow z = \left(\frac{p}{3}\right)^{\frac{3}{2}} + \left(\frac{q}{3}\right)^{\frac{3}{2}} \text{ which is the required p.d.e.}$$

13. $z = (x+a)^3 + (y-b)^2$ — Ans: $z = \left(\frac{p}{3}\right)^{\frac{3}{2}} + \left(\frac{q}{2}\right)^2$

14. $(x-a)^2 + (y-b)^2 = z^2 \cot^2 \alpha$

    Solution:

    Given $(x-a)^2 + (y-b)^2 = z^2 \cot^2 \alpha$...........(1) — no.of a.c=no.of I.V

    Differentiate (1) par. w.r.t x and y,

$$2(x-a) = 2zp\cot^2 \alpha, \qquad 2(y-b) = 2zq\cot^2 \alpha$$
$$(x-a) = zp\cot^2 \alpha, \qquad (y-b) = zq\cot^2 \alpha$$

    Sub in (1)

$$[zp\cot^2 \alpha]^2 + [zq\cot^2 \alpha]^2 = z^2 \cot^2 \alpha$$
$$z^2 \cot^4 \alpha(p^2 + q^2) = z^2 \cot^2 \alpha$$
$$\Rightarrow p^2 + q^2 = \tan^2 \alpha \text{ which is the required p.d.e.}$$

15. Find the PDE of the family of spheres having centres on the z-axis.

    Solution:

    Given that the sphere lies on the z-axis.

    Centre is $(0, 0, c)$. Let Radius $= r$.

    $\therefore$ Eqn of the sphere

$$x^2 + y^2 + (z-c)^2 = r^2\text{.........(1)}$$

    no.of a.c=no.of I.V

    Differentiate (1) par. w.r.t x and y,

$$2x + 2(z-c)\frac{\partial z}{\partial x} = 0, \qquad 2y + 2(z-c)\frac{\partial z}{\partial y} = 0$$

$$x + (z-c)p = 0, \qquad y + (z-c)q = 0$$

$$(z-c)p = -x.........(2), \qquad (z-c)q = -y. .......(3)$$

$$\frac{Eqn(2)}{Eqn(3)} \Rightarrow \frac{p}{q} = \frac{-x}{-y}$$

$\Rightarrow py = xq$ which is the required p.d.e.

16. Find the PDE of all plane having equal intercept on the x and y axis.

    Solution:

    Intercept form of the plane eqn is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

    Given :$a = b$

    $$\Rightarrow \frac{x}{a} + \frac{y}{a} + \frac{z}{c} = 1. .......(1)$$

    no.of a.c=no.of I.V

    Differentiate (1) par. w.r.t x and y,

    $$\frac{1}{a} + 0 + \frac{1}{c}\frac{\partial z}{\partial x} = 0, \qquad 0 + \frac{1}{a} + \frac{1}{c}\frac{\partial z}{\partial y} = 0$$

    $$\frac{1}{a} + \frac{1}{c}p = 0 , \qquad \frac{1}{a} + \frac{1}{c}q = 0$$

    $$\frac{1}{a} = -\frac{1}{c}p.......(2) \qquad \frac{1}{a} = -\frac{1}{c}q.......(3)$$

    From (2) and (3)

    $$-\frac{1}{c}p = -\frac{1}{c}q$$

    $\Rightarrow p = q$ which is the required p.d.e.

17. Form the PDE of the family of sphere having their centres on the line $x = y = z$

    Hint:$(x-a)^2 + (y-a)^2 + (z-a)^2 = r^2$

    Ans : $(y-z)p + (z-x)q = x - y$

## Formation of PDE by eliminating arbitrary function:

Form the PDE by eliminating the arbitrary functions

1. $z = f\left(\frac{x}{y}\right)$

   Solution:

   Given $z = f\left(\frac{x}{y}\right)$

   Diff par w.r.t. x and y,

   $$\frac{\partial z}{\partial x} = f'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right), \qquad \frac{\partial z}{\partial y} = f'\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right)$$

   $$p = f'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right).......(1) \qquad q = f'\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right) ......(2)$$

   $$\frac{Eqn(1)}{Eqn(2)} \Rightarrow \frac{p}{q} = \frac{\frac{1}{y}}{\frac{-x}{y^2}}$$

   $$\frac{p}{q} = \frac{1}{y}\left(\frac{-y^2}{x}\right)$$

   $$\frac{p}{q} = \frac{-y}{x}$$

   $\Rightarrow px = -qy$

   $\therefore px + qy = 0$ which is the required p.d.e.

2. $z = f\left(\frac{y}{x}\right)$ **Ans : $xp + yq = 0$**

3. $z = f(xy)$ **Ans : $xp = yq$**

4. $z = f(x^2 + y^2)$ Solution:

   Given $z = f(x^2 + y^2)$

   Diff par w.r.t. x & y,

   $$p = \frac{\partial z}{\partial x} = f'(x^2 + y^2)2x.......(1), \qquad q = \frac{\partial z}{\partial y} = f'(x^2 + y^2)2y........(2)$$

   $$\frac{Eqn(1)}{Eqn(2)} \Rightarrow \frac{p}{q} = \frac{2x}{2y}$$

   $\therefore py = qx$ which is the required p.d.e.

5. $z^2 - xy = f\left(\frac{x}{z}\right)$

   Solution:

   Let $\varphi\left(z^2 - xy, \frac{x}{z}\right) = 0$

   where $u = z^2 - xy$ and $v = \frac{x}{z}$

   Hence the required PDE is of the form $Pp + Qq = R$

   Where

   $$\begin{aligned} P &= \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y} \\ &= (-x)\left(-\frac{x}{z^2}\right) - 2z(0) \\ P &= \frac{x^2}{z^2} \\ Q &= \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z} \\ &= (2z)\left(\frac{1}{z}\right) - (-y)\left(-\frac{x}{z^2}\right) \\ Q &= 2 - \frac{xy}{z^2} \\ R &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \\ &= (-y)(0) - (-x)\left(\frac{1}{z}\right) \\ R &= \frac{x}{z} \end{aligned}$$

   The required equation is

   $$\begin{aligned} \left(\frac{x^2}{z^2}\right)p + \left(2 - \frac{xy}{z^2}\right)q &= \frac{x}{z} \\ \left(\frac{x^2}{z^2}\right)p + \left(\frac{2z^2 - xy}{z^2}\right)q &= \frac{x}{z} \\ x^2p + (2z^2 - xy)q &= xz \text{ which is the required p.d.e.} \end{aligned}$$

6. $\varphi(x^2 - y^2, z) = 0$ **Ans: $yp + xq = 0$**

7. $\varphi(x^2 + y^2 + z^2, ax + by + cz) = 0$ **Ans: $(bz - cy)p + (cx - az)q = ay - bx$**

8. $\varphi(x^2+y^2+z^2, x+y+z)=0$ 

Ans: $(z-y)p+(x-z)q=y-x$

**Two Functions[use p,q,r,s,t]**

9. $z=g(y+x)+xf(y+x)$

Solution:

Given $z=xf(y+x)+g(y+x)$. ...(1)

Diff par w.r.t. x & y,

$$\begin{aligned} p &= f(y+x)+xf'(y+x)+g'(y+x). \text{ ...(2)} \\ q &= xf'(y+x)+g'(y+x). \text{ ...(3)} \end{aligned}$$

Again diff (2) and (3) w.r.t x and y,

$$\begin{aligned} r &= f'(y+x)+xf''(y+x)+f'(y+x)+g''(y+x) \\ t &= xf''(y+x)+g''(y+x) \\ s &= f'(y+x)+xf''(y+x)+g''(y+x) \\ \text{Now } r+t &= 2[f'(y+x)+xf''(y+x)+g''(y+x)] \\ r+t &= 2s \end{aligned}$$

10. $z=f(x+ct)+g(x-ct)$

Solution:

Given $z=f(x+ct)+g(x-ct)$

Diff par w.r.t x & t,

$$\begin{aligned} \frac{\partial z}{\partial x} &= f'(x+ct)+g'(x-ct) \\ \frac{\partial z}{\partial t} &= f'(x+ct)c+g'(x-ct)(-c) \end{aligned}$$

Again diff par w.r.t x & t,

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= f''(x+ct)+g''(x-ct) \\ \frac{\partial^2 z}{\partial t^2} &= f''(x+ct)c^2+g''(x-ct)(-c)^2 \\ &= c^2[f''(x+ct)+g''(x-ct)] \\ \frac{\partial^2 z}{\partial t^2} &= c^2\frac{\partial^2 z}{\partial x^2} \end{aligned}$$

11. $z=x^2f(y)+y^2g(x)$

Solution:

Given $z=x^2f(y)+y^2g(x)$. ....(1)

Diff par w.r.t. x & y,

$$\begin{aligned} p &= 2xf(y) + y^2g'(x). \ ...(2) \\ q &= x^2f'(y) + 2yg(x). \ ...(3) \end{aligned}$$

Again diff (2) and (3) w.r.t x and y,

$$\begin{aligned} r &= 2f(y) + y^2g''(x). \ ...(4) \\ t &= x^2f''(y) + 2g(y). \ ...(5) \\ s &= 2xf'(y) + 2yg'(x)..(6) \end{aligned}$$

$$\begin{aligned} (2) \times x \Rightarrow px &= 2x^2f(y) + xy^2g'(x) \\ (3) \times y \Rightarrow qy &= x^2yf'(y) + 2y^2g(x) \\ px + qy &= 2[x^2f(y) + y^2g(x)] + xy[xf'(y) + yg'(x)] \\ &= 2z + xy\left[\frac{s}{2}\right] \\ 2(px + qy) &= 4z + xys \end{aligned}$$

12. $z = f(x^3 + 2y) + g(x^3 - 2y)$ Ans: $4xr = 9x^5t + 8p$

## TYPES OF SOLUTION

### Complete solution (or) Complete Integral

A solution in which the number of arbitrary constants is equal to the number of independent variables.

### Particular Integral

In complete integral if we give particular values to the arbitrary constants.

### Singular Integral

Let $f(x, y, z, p, q) = 0$ be a PDE whose complete integration is $\phi(x, y, z, a, b) = 0. \ ...(1)$

Diff (1) partially w.r.t a and b and then equal to zero, we get

$$\begin{aligned} \frac{\partial \phi}{\partial \phi} &= 0. \ ...(2) \\ \frac{\partial \phi}{\partial b} &= 0.....(3) \end{aligned}$$

Eliminating a and b by using eqn(1),(2) and (3).

The eliminant of a and b is called **Singular Integral**.

**Type 1: Form $F(p, q) = 0$**

**There is no Singular Integral for Type 1**

1. Find the Complete integral of $\sqrt{p} + \sqrt{q} = 1$

   Solution:

   Given $\sqrt{p} + \sqrt{q} = 1$. ...(1)

   This is type 1,

   The complete integral is $z = ax + by + c$

   put $p = a$ and $q = b$ in (1)

$$\begin{aligned} \sqrt{a} + \sqrt{b} &= 1 \\ \sqrt{b} &= 1 - \sqrt{a} \\ b &= (1 - \sqrt{a})^2 \end{aligned}$$

   $\Rightarrow$ the complete integral is $z = ax + (1 - \sqrt{a})^2 y + c,$ where a and c are arbitray constants.

2. Find the Complete integral of $p + q = pq$

   Solution:

   Given $p + q = pq$.....(1)

   This is type 1,

   The complete integral is $z = ax + by + c$

   put $p = a$ and $q = b$ in (1)

$$\begin{aligned} a + b &= ab \\ b &= ab - a \\ b &= a(b - 1) \end{aligned}$$

   $\Rightarrow$ the complete integral is $z = ax + a(b - 1)y + c,$ where a and c are arbitray constants.

3. Find the Complete integral of $p^2 + q^2 = npq$

   Solution:

   Given $p^2 + q^2 = npq$.....(1)

   This is type 1,

   The complete integral is $z = ax + by + c$

   put $p = a$ and $q = b$ in (1)

$$\begin{aligned} a^2 + b^2 &= nab \\ b^2 - nab + a^2 &= 0 \end{aligned}$$

$$\Rightarrow b = \frac{na \pm \sqrt{n^2a^2 - 4a^2}}{2}$$
$$= \frac{na \pm a\sqrt{n^2 - 4}}{2}$$

$\Rightarrow$ the complete integral is $z = ax + \left.\frac{n \pm \sqrt{n^2 - 4}}{2}\right) ay + c,$ where a and c are arbitray constants.

**Type 2: Form $z = px + qy + f(p, q)$[Clairaut's Form]**

1. Solve the equation $z = px + qy + p^2 - q^2$

   Solution:

   Given $z = px + qy + p^2 - q^2$

   This is type 2[Clairaut's Form],

   put $p = a$ and $q = b$ in (1)

   $\therefore$ The complete integral is $z = ax + by + a^2 - b^2$. ...(1)

   **To find Singular Integral**

   Diff (1) par w.r.t a and b, we get

$$0 = x + 2a \Rightarrow a = -\frac{x}{2}$$
$$0 = y - 2b \Rightarrow b = \frac{y}{2}$$

   substituting in (1), we get

$$z = -\frac{x}{2}x + \frac{y}{2}y + \left(\frac{-x}{2}\right)^2 - \left(\frac{y}{2}\right)^2$$
$$= -\frac{x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4}$$
$$= -\frac{x^2}{4} + \frac{y^2}{4}$$
$$4z = y^2 - x^2, \quad \text{which is the singular solution}$$

2. solve $z = px + qy + p^2q^2$

   Solution:

   Given $z = px + qy + p^2q^2$

   This is type 2[Clairaut's Form],

   put $p = a$ and $q = b$ in (1)

   $\therefore$ The complete integral is $z = ax + by + a^2b^2$. ........(1)

   **To find Singular Integral**

   Diff (1) par w.r.t a and b, we get

$$\frac{\partial z}{\partial a} = x + 2ab^2 = 0$$

$$\begin{aligned}
\frac{\partial z}{\partial b} &= y + 2ba^2 = 0 \\
\Rightarrow x &= -2ab^2. \text{........}(2) \\
y &= -2ba^2. \text{........}(3) \\
(2) \Rightarrow a &= -\frac{x}{2b^2}. \text{........}(4)
\end{aligned}$$

Substitute (4) in (3)

$$\begin{aligned}
y &= -2b\left(-\frac{x}{2b^2}\right)^2 \\
&= -2b\left(\frac{x^2}{4b^4}\right) \\
&= -\frac{x^2}{2b^3} \\
b^3 &= -\frac{x^2}{2y} \\
b &= \left(-\frac{x^2}{2y}\right)^{\frac{1}{3}} \text{.........}(5)
\end{aligned}$$

Substitute (5) in (4)

$$\begin{aligned}
a &= -\frac{x}{2\left(-\frac{x^2}{2y}\right)^{\frac{2}{3}}} \\
&= -\frac{x}{2\left(\frac{x^{\frac{4}{3}}}{2^{\frac{2}{3}}y^{\frac{2}{3}}}\right)} \\
&= -\frac{x \times 2^{\frac{2}{3}}y^{\frac{2}{3}}}{2x^{\frac{4}{3}}} \\
&= -\left(\frac{y^2}{2x}\right)^{\frac{1}{3}} \text{.........}(6)
\end{aligned}$$

Substitute (5) and (6) in (1)

$$\begin{aligned}
z &= -x\left(\frac{y^2}{2x}\right)^{\frac{1}{3}} - y\left(\frac{x^2}{2y}\right)^{\frac{1}{3}} + \left(\frac{y^2}{2x}\right)^{\frac{2}{3}}\left(\frac{x^2}{2y}\right)^{\frac{2}{3}} \\
&= -x\left(\frac{y^{\frac{2}{3}}}{2^{\frac{1}{3}}x^{\frac{1}{3}}}\right) - y\left(\frac{x^{\frac{2}{3}}}{2^{\frac{1}{3}}y^{\frac{1}{3}}}\right) + \left(\frac{y^2}{2x} \times \frac{x^2}{2y}\right)^{\frac{2}{3}} \\
&= -\left(\frac{x^{\frac{2}{3}}y^{\frac{2}{3}}}{2^{\frac{1}{3}}}\right) - \left(\frac{x^{\frac{2}{3}}y^{\frac{2}{3}}}{2^{\frac{1}{3}}}\right) + \left(\frac{x^{\frac{2}{3}}y^{\frac{2}{3}}}{4^{\frac{2}{3}}}\right) \\
z^3 &= -\frac{27}{16}x^2y^2, \text{which is the singular solution}
\end{aligned}$$

3. Find the singular solution of $z = px + qy + \sqrt{1 + p^2 + q^2}$

   Solution:

   Given $z = px + qy + \sqrt{1 + p^2 + q^2}$

   This is type 2[Clairaut's Form],

put $p = a$ and $q = b$ in (1)

$\therefore$ The complete integral is $z = ax + by + \sqrt{1+a^2+b^2}$..........(1)

**To find Singular Integral**

Diff (1) par w.r.t a and b, we get

$$\begin{aligned}
\frac{\partial z}{\partial a} = x + \frac{1}{2\sqrt{1+a^2+b^2}}(2a) &= 0 \\
x + \frac{a}{\sqrt{1+a^2+b^2}} &= 0 \\
x &= -\frac{a}{\sqrt{1+a^2+b^2}}. \text{........(2)} \\
\frac{\partial z}{\partial b} = x + \frac{1}{2\sqrt{1+a^2+b^2}}(2b) &= 0 \\
y + \frac{b}{\sqrt{1+a^2+b^2}} &= 0 \\
y &= -\frac{b}{\sqrt{1+a^2+b^2}}. \text{........(3)} \\
\Rightarrow x^2 &= \frac{a^2}{1+a^2+b^2} \\
y^2 &= \frac{b^2}{1+a^2+b^2} \\
\text{Now} \qquad x^2+y^2 &= \frac{a^2+b^2}{1+a^2+b^2} \\
1-(x^2+y^2) &= 1 - \frac{a^2+b^2}{1+a^2+b^2} \\
1-x^2-y^2 &= \frac{1+a^2+b^2-a^2-b^2}{1+a^2+b^2} \\
&= \frac{1}{1+a^2+b^2} \\
1+a^2+b^2 &= \frac{1}{1-x^2-y^2} \\
\sqrt{1+a^2+b^2} &= \frac{1}{\sqrt{1-x^2-y^2}}. \text{.......(4)}
\end{aligned}$$

From (2) and (3)

$$\frac{a}{\sqrt{1+a^2+b^2}} = -x \qquad\qquad \frac{b}{\sqrt{1+a^2+b^2}} = -y$$

$$a = -x\sqrt{1+a^2+b^2} \qquad\qquad b = (-y)\sqrt{1+a^2+b^2}$$

$$a = \frac{-x}{\sqrt{1-x^2-y^2}} \qquad\qquad a = \frac{-x}{\sqrt{1-x^2-y^2}} \quad \text{by (4)}$$

substituting in (1)

$$\begin{aligned}
z &= \frac{-x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}} \\
&= \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}} \\
z &= \sqrt{1-x^2-y^2} \\
z^2 &= 1-x^2-y^2 \\
\therefore x^2+y^2+z^2 &= 1 \text{which is the singular solution.}
\end{aligned}$$

4. solve $z = px + qy + 2\sqrt{pq}$ — Ans: $xy = 1$

5. solve $z = px + qy + \frac{p}{q} - p$ — Ans: $z = \frac{y}{1-x}$

6. solve $z = px + qy + p^2 - q^2$ — Ans: $y^2 - x^2 = 4z$

7. Find the complete integral of $(z - px - qy)(p + q) = 1$ — Ans: $z = ax + by + \frac{1}{a+b}$

**Type 3: Form $f(p, q, z) = 0$ or $f(p, q, x) = 0$ or $f(p, q, y) = 0$**

1. Solve $p(1 + q) = qz$

   Solution:

   Given $p(1 + q) = qz$..........(1)

   This is type 3,$[f(p, q, z) = 0]$

   Put $q = ap$

   substitute in (1)

$$\begin{aligned} p(1+ap) &= apz \\ 1+ap &= az \\ ap &= az-1 \\ p &= \frac{az-1}{a} \qquad\qquad q = ap = az - 1 \end{aligned}$$

We have $dz = pdx + qdy$,

$$\begin{aligned} \int dz &= \int \frac{az-1}{a}dx + \int (az-1)dy \\ \int \frac{dz}{az-1} &= \int \frac{1}{a}dx + \int dy \\ \frac{1}{a}\int \frac{adz}{az-1} &= \frac{1}{a}\int dx + \int dy \\ \frac{1}{a}\log(az-1) &= \frac{1}{a}x + y + c_1 \\ \log(az-1) &= x + ay + c. \text{........(2)} \end{aligned}$$

which is a complete integral, where a and c are constants.

**To find the Singular Integral**

Diff (2) par w.r.t a and c

when diff par w.r.t c, we get $0 = 1$ which is not true.

So there is no Singular integral.

**To find the General integral**

Put $c = \phi(a)$ in (2)

$\Rightarrow \log(az - 1) = x + ay + \phi(a)$. ........(3)

Diff (3) w.r.t a and Eliminating a, we get the General Integral

2. Solve $z^2(p^2+q^2+1)=1$

Solution:

Given $z^2(p^2+q^2+1)=1$..........(1)

This is type 3,$[f(p,q,z)=0]$ Put $q=ap$

substitute in (1)

$$\begin{aligned} z^2(p^2+a^2p^2+1) &= 1 \\ p^2z^2(a^2+1)+z^2 &= 1 \\ p^2 &= \frac{1-z^2}{z^2a^2+1} \\ p &= \frac{\sqrt{1-z^2}}{\sqrt{z^2(a^2+1)}} \end{aligned} \qquad q=ap=a\frac{\sqrt{1-z^2}}{\sqrt{z^2(a^2+1)}}$$

$$\text{We have } dz = pdx+qdy$$

$$\begin{aligned} \int dz &= \int \frac{\sqrt{1-z^2}}{z\sqrt{a^2+1}}dx + \int a\frac{\sqrt{1-z^2}}{z\sqrt{a^2+1}}dy \\ \int \frac{z}{\sqrt{1-z^2}}dz &= \int \frac{1}{\sqrt{a^2+1}}dx + \int a\frac{1}{\sqrt{a^2+1}}dy. ...(i) \end{aligned}$$

$$\begin{aligned} \text{put } 1-z^2 &= t^2 \\ -2zdz &= 2tdt \\ -zdz &= tdt \end{aligned}$$

sub in (i)

$$\begin{aligned} \int \frac{-tdt}{t} &= \frac{1}{\sqrt{a^2+1}}\int dx + a\frac{1}{\sqrt{a^2+1}}\int dy. ...(i) \\ \sqrt{1+a^2}(-t) &= x+ay+c \\ -\sqrt{1+a^2}\sqrt{1-z^2} &= x+ay+c. .........(2) \end{aligned}$$

which is a complete integral, where a and c are constants.

**To find the Singular Integral**

Diff (2) par w.r.t a and c

when diff par w.r.t c, we get $0=1$ which is not true.

So there is no Singular integral.

**To find the General integral**

Put $c=\phi(a)$ in (2)

$\Rightarrow -\sqrt{1+a^2}\sqrt{1-z^2}=x+ay+\phi(a).$ ........(3)

Diff (3) w.r.t a and Eliminating a, we get the General Integral.

3. $p(1-q^2) = q(1-z)$

Solution:

Given $p(1-q^2) = q(1-z)$ ..........(1)

This is type 3,$[f(p,q,z) = 0]$

Put $q = ap$

substitute in (1)

$$\begin{aligned}
p(1-a^2p^2) &= ap(1-z) \\
1-a^2p^2 &= a-az \\
a^2p^2 &= 1-a+az \\
p^2 &= \frac{1-a+az}{a^2} \\
p &= \frac{\sqrt{1-a+az}}{a} \\
\text{We have } dz &= pdx + qdy
\end{aligned}$$

$$\begin{aligned}
\int dz &= \int \frac{\sqrt{1-a+az}}{a} dx + \int a \frac{\sqrt{1-a+az}}{a} dy \\
\int \frac{dz}{\sqrt{1-a+az}} &= \frac{1}{a}\int dx + \int dy. \text{ ........(2)}
\end{aligned}$$

$$\begin{aligned}
\text{put } 1-a+az &= t^2 \\
adz &= 2tdt
\end{aligned}$$

sub in (2)

$$\begin{aligned}
\int \frac{2tdt}{at} &= \frac{1}{a}\int dx + \int dy \\
\frac{2t}{a} &= \frac{1}{a}x + y + c \\
2\sqrt{1-a+az} &= x + ay + c. \text{ ........(3)}
\end{aligned}$$

which is a complete integral, where a and c are constants.

**To find the Singular Integral**

Diff (2) par w.r.t a and c

when diff par w.r.t c, we get $0 = 1$ which is not true.

So there is no Singular integral.

**To find the General integral**

Put $c = \phi(a)$ in (2)

$$\Rightarrow 2\sqrt{1-a+az} = x+ay+\phi(a). \quad ........(3)$$

Diff (3) w.r.t a and Eliminating a, we get the General Integral.

4. $p(1+q^2) = q(z-a)$ — Ans: $2\sqrt{bz-(ab+1)} = x+by+c$

5. Solve $9(p^2z+q^2) = 4$ — Ans: $(z+a^2)^{\frac{3}{2}} = x+ay+c$

6. Solve $z^2 = 1+p^2+q^2$ — Ans: $\cos h^{-1}z = \frac{1}{\sqrt{1+a^2}}(x+ay)+c$

**Type 4: Method of Separable[Form: $f_1(x,p) = f_2(y,q)$]**

1. Solve $p^2y(1+x^2) = qx^2$

   Solution:

   Given $p^2y(1+x^2) = qx^2$ . ...(1)

   This is type 4,$[f_1(x,p) = f_2(y,q)]$

$$\frac{p^2(1+x^2)}{x^2} = \frac{q}{y} = a$$

$$\frac{p^2(1+x^2)}{x^2} = a \qquad \frac{q}{y} = a$$

$$p^2 = \frac{ax^2}{1+x^2} \qquad q = ya$$

$$p = \frac{x\sqrt{a}}{\sqrt{1+x^2}} \qquad q = ya$$

$$\begin{aligned}
\text{We have } dz &= pdx+qdy \\
\int dz &= \int \frac{x\sqrt{a}}{\sqrt{1+x^2}}dx + \int yady. \quad .........(1) \\
\text{put } 1+x^2 &= t^2 \\
2xdx &= 2tdt \Rightarrow xdx = tdt \\
\text{sub in (1)} & \\
\int dz &= \sqrt{a}\int \frac{tdt}{t} + a\int ydy \\
z &= \sqrt{a}t + a\frac{y^2}{2} + c \\
z &= \sqrt{a}\sqrt{1+x^2} + a\frac{y^2}{2} + c \\
z &= \sqrt{a(1+x^2)} + \frac{ay^2}{2} + c. \quad .........(2)
\end{aligned}$$

which is a complete integral, where a and c are constants

**To find the Singular Integral**

Diff (2) par w.r.t a and c

when diff par w.r.t c, we get $0 = 1$ which is not true.

So there is no Singular integral.

**To find the General integral**

Put $c = \phi(a)$ in (2)

$$\Rightarrow z = \sqrt{a(1+x^2)} + \frac{ay^2}{2} + \phi(a). .......(3)$$

Diff (3) w.r.t a and Eliminating a, we get the General Integral.

2. Solve $p^2 + q^2 = x + y$

Solution:

Given $p^2 + q^2 = x + y$.....(1)

This is type 4,$[f_1(x, p) = f_2(y, q)]$

$$p^2 - x = y - q^2 = a$$

$$p^2 - x = a \qquad y - q^2 = a$$

$$p^2 = a + x \qquad q^2 = y - a$$

$$p = \sqrt{a + x} \qquad q = \sqrt{y - a}$$

$$\begin{aligned} \text{We have } dz &= pdx + qdy \\ \int dz &= \int \sqrt{x+a}dx + \int \sqrt{y-a}dy \\ z &= \frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{(y-a)^{\frac{3}{2}}}{\frac{3}{2}} + c \\ z &= \frac{2}{3}(x+a)^{\frac{3}{2}} + \frac{2}{3}(y-a)^{\frac{3}{2}} + c. ...(2) \end{aligned}$$

which is a complete integral, where a and c are constants

**To find the Singular Integral**

Diff (2) par w.r.t a and c

when diff par w.r.t c, we get $0 = 1$ which is not true.

So there is no Singular integral.

**To find the General integral**

Put $c = \phi(a)$ in (2)

$$\Rightarrow z = \frac{2}{3}(x+a)^{\frac{3}{2}} + \frac{2}{3}(y-a)^{\frac{3}{2}} + \phi(a). ...(3)$$

Diff (3) w.r.t a and Eliminating a, we get the General Integral.

3. Solve $q = 2px$

Solution:

Given $q = 2px$.....(1)

This is type 4,$[f_1(x, p) = f_2(y, q)]$

$$q = 2px = a$$

$$2px = a \qquad\qquad y = a$$
$$p = \frac{a}{2x} \qquad\qquad q = a$$

$$\begin{aligned} \text{We have } dz &= pdx + qdy \\ \int dz &= \int \frac{a}{2x}dx + \int ady \\ z &= \frac{a}{2}\log x + ay + c. \ ...(2) \end{aligned}$$

which is a complete integral, where a and c are constants

**To find the Singular Integral**

Diff (2) par w.r.t a and c

when diff par w.r.t c, we get $0 = 1$ which is not true.

So there is no Singular integral.

**To find the General integral**

Put $c = \phi(a)$ in (2)

$\Rightarrow z = \frac{a}{2}\log x + ay + \phi(a). \ ...(3)$

Diff (3) w.r.t a and Eliminating a, we get the General Integral.

4. Find the complete integral of $pq = xy$ 

   Ans: $z = a\frac{x^2}{2} + \frac{1}{a}\frac{y^2}{2} + c$

5. Find the complete integral of $p + q = x + y$

   Ans: $z = \frac{(x+a)^2}{2} + \frac{(y-a)^2}{2} + c$

6. Find the complete integral of $p^2 + q^2 = x^2 + y^2$

   Hint: $\int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sin h^{-1}\frac{x}{a}$

   $\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cos h^{-1}\frac{x}{a}$

   Ans: $z = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sin h^{-1}\frac{x}{a} + \frac{y}{2}\sqrt{y^2-a^2} - \frac{a^2}{2}\cos h^{-1}\frac{y}{a} + c$

## Solving second and higher order with constant coefficients homogeneous and non homogeneous differential equation

In this chapter z will always represent a function of x and y, i.e $z = f(x, y)$

Notations:

$$D = \frac{\partial}{\partial x}, D' = \frac{\partial}{\partial y}, p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y} \& t = \frac{\partial^2 z}{\partial y^2}$$

Definition:

A linear p.d.e with constant coefficient in which all the partial derivative are of the same order is called homogeneous, otherwise it is called non-homogeneous

(HOMOGENEOUS)Form: $f(D, D')z = F(x, y)$

The solution is $z = C.F + P.I$

To find C.F

Put $D = m$ and $D' = 1$

| Roots | C.F |
|---|---|
| real(imaginary) and Different $m_1 \neq m_2$ | $f_1(y+m_1x)+f_2(y+m_2x)$ |
| real(imaginary) and same $m_1 = m_2 = m$ | $f_1(y+mx)+xf_2(y+mx)$ |

To find P.I

| Form | P.I | Substitution |
|---|---|---|
| $f(D,D') = e^{ax+by}$ | $\frac{1}{f(D,D')}e^{ax+by}$ <br> $= \frac{1}{f(a,b)}e^{ax+by}$ | $D \to a$, $D' \to b$ |
| $f(D,D') = \sin(ax+by)$ or $\cos(ax+by)$ | $\frac{1}{f(D,D')}\sin(ax+by)$ or $\cos(ax+by)$ | $D^2 \to -a^2$, $D'^2 \to -b^2$, $DD' \to -ab$ |
| $f(D,D') = x^m y^n$ | $\frac{1}{f(D,D')}x^m y^n$ <br> $= [f(D,D')]^{-1}x^m y^n$ | Binomial theorem |
| $f(D,D') = e^{ax+by}\sin(cx+dy)$ | $\frac{1}{f(D,D')}e^{ax+by}\sin(cx+dy)$ <br> $= e^{ax+by}\frac{1}{f(D+a,D'+b)}\sin(cx+dy)$ | $D \to D+a$, $D' \to D'+b$ <br> Solve by form 2 |
| $f(D,D') = \sin(ax+by)\phi(x,y)$ Or $e^{ax+by}\phi(x,y)$ | $\frac{1}{f(D,D')}F(x,y)$ <br> $= \frac{1}{(D-m_1D')(D-m_2D')}F(x,y)$ <br> $= \frac{1}{(D-m_1D')}\int F(x,c-m_2x)dx$ | $y = c - m_2x$ |

**Note:**

If the Dr. is zero then put x in Nr. and differentiate w.r.t D in Dr.

1. Solve $(D^3 - 7DD'^2 - 6D'^3)z = e^{2x+y} + \sin(x+2y)$

Solution:

A.E : Put $D = m$ and $D' = 1$

i.e $m^3 - 7m - 6 = 0$

$$\begin{array}{r|rrrr} & 1 & 0 & -7 & -6 \\ -1 & & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

i.e $m^2 - m - 6 = 0$

$(m+2)(m-3) = 0$

$\Rightarrow m = -2, 3$

$\therefore m = -1, -2, 3$

C.F $= f_1(y-x) + f_2(y-2x) + f_3(y+3x)$

To find P.I

Ist,

$D = a = 2$

$D' = b = 1$

IInd,

$D^2 = -a^2 = -1$

$D'^2 = -b^2 = -4$

$D'^2 = -b^2 = -4$

$$\begin{aligned}
P.I &= \frac{1}{D^3 - 7DD'^2 - 6D'^3} e^{2x+y} + \frac{1}{D^3 - 7DD'^2 - 6D'^3} \sin(x+2y) \\
&= \frac{1}{8 - 7 \times 2 \times 1 - 6} e^{2x+y} + \frac{1}{-D + 28D + 24D'} \sin(x+2y) \\
&= \frac{1}{-12} e^{2x+y} + \frac{1}{27D + 24D'} \sin(x+2y) \\
&= \frac{-1}{12} e^{2x+y} + \frac{D}{27D^2 + 24DD'} \sin(x+2y) \\
&= \frac{-e^{2x+y}}{12} + \frac{D}{27(-1) + 24(-2)} \sin(x+2y) \\
&= \frac{-e^{2x+y}}{12} + \frac{\cos(x+2y)}{-27 - 48} \\
&= \frac{-e^{2x+y}}{12} + \frac{\cos(x+2y)}{-75} \\
&= \frac{-e^{2x+y}}{12} - \frac{\cos(x+2y)}{75} \\
\therefore z &= C.F + P.I \\
z &= f_1(y-x) + f_2(y-2x) + f_3(y+3x) - \frac{e^{2x+y}}{12} - \frac{\cos(x+2y)}{75}
\end{aligned}$$

2. Solve $(D^2 - D'^2)z = e^{x+2y} + \sin(2x - y)$

Ans: $z = f_1(y+x) + f_2(y-x) - \frac{e^{x+2y}}{3} - \frac{\sin(2x-y)}{3}$

3. Solve $(D^3 + D^2D' + 4DD'^2 + 4D'^3)z = \cos(2x+y)$

Ans: $z = f_1(y-x) + f_2(y+2ix) + f_3(y-2ix) - \frac{1}{24}\sin(2x+y)$

4. Solve $(D^3 - 7DD'^2 - 6D'^3)z = \cos(x+y)$

Ans: $z = f_1(y-x) + f_2(y-2x) + f_3(y+3x) + \frac{1}{12}\sin(x+y)$

5. Solve $(D^2 + DD' - 6D'^2)z = \cos(x+2y)$

Ans: $z = f_1(y+2x) + f_2(y-3x) + \frac{1}{21}\cos(x+2y)$

6. Solve $(D^2 + 3DD' - 4D'^2)z = \sin y$

   Ans: $z = f_1(y - 4x) + f_2(y + x) + \frac{1}{4}\sin y$

7. Solve $(D^2 - 2DD' + D'^2)z = e^{x+2y} + \sin(2x - 3y)$

   Ans: $z = f_1(y + x) + xf_2(y + x) + e^{x+2y} - \frac{1}{25}\sin(2x - 3y)$

8. Solve $(D^2 + 2DD' + D'^2)z = e^{x+2y} + \sin(2x - y)$

   Ans: $z = f_1(y - x) + xf_2(y - x) + \frac{1}{9}e^{x+2y} - \sin(2x - y)$

9. Solve $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = e^{x+2y} + 4\sin(x + y)$

   Ans: $z = f_1(y) + xf_2(y) + f_3(y + 2x) - \frac{1}{3}e^{x+2y} - 4\cos(x + y)$

10. Solve $(4D^2 - 4DD' + D'^2)z = e^{3x-2y} + \sin x$

    Ans: $z = f_1\left(y + \frac{1}{2}x\right) + xf_2\left(y + \frac{1}{2}x\right) + \frac{1}{64}e^{3x-2y} - \frac{1}{4}\sin x$

11. Solve $(D^2 - DD' - 20D'^2)z = e^{5x+y} + \sin(4x - y)$

    Solution:

    A.E : Put $D = m$ and $D' = 1$

    $$\text{i.e } m^2 - m - 20 = 0$$
    $$(m - 5)(m + 4) = 0$$
    $$\Rightarrow m = 5, -4$$

    $\therefore$ C.F $= f_1(y + 5x) + f_2(y - 4x)$

    To find P.I

    $$\begin{aligned}
    P.I &= \frac{1}{D^2 - DD' - 20D'^2}e^{5x+y} + \frac{1}{D^2 - DD' - 20D'^2}\sin(4x - y) \\
    &= \frac{1}{25 - 5 - 20}e^{5x+y} + \frac{1}{-16 - 4 + 20}\sin(4x - y) \\
    &= \frac{1}{0}e^{5x+y} + \frac{1}{0}\sin(4x - y) \\
    &= \frac{x}{2D - D'}e^{5x+y} + \frac{x}{2D - D'}\sin(4x - y) \\
    &= \frac{x}{10 - 1}e^{5x+y} + x\frac{D}{2D^2 + DD'}\sin(4x - y) \\
    &= \frac{x}{9}e^{5x+y} + x\frac{\cos(4x - y)2}{2(-16) - 4} \\
    &= \frac{x}{9}e^{5x+y} + x\frac{4\cos(4x - y)2}{-36} \\
    &= \frac{x}{9}e^{5x+y} - \frac{x}{9}\cos(4x - y) \\
    \therefore z &= C.F + P.I \\
    z &= f_1(y + 5x) + f_2(y - 4x) + \frac{x}{9}e^{5x+y} - \frac{x}{9}\cos(4x - y)
    \end{aligned}$$

    | |
    |---|
    | Ist, |
    | $D = a = 5$ |
    | $D' = b = 1$ |
    | IInd, |
    | $D^2 = -a^2 = -16$ |
    | $D'^2 = -b^2 = -1$ |
    | $D'^2 = -b^2 = 4$ |

12. Solve $(D^2 - DD' - 30D'^2)z = xy + e^{6x+y}$

    Solution:

    A.E : Put $D = m$ and $D' = 1$

i.e $m^2 - m - 30 = 0$

$(m-6)(m+5) = 0$

$\Rightarrow m = 6, -5$

$\therefore$ C.F $= f_1(y+6x) + f_2(y-5x)$

To find P.I

| $D = a = 6$ |
|---|
| $D' = b = 1$ |

$$
\begin{aligned}
P.I &= \frac{1}{D^2 - DD' - 30D'^2} e^{6x+y} + \frac{1}{D^2 - DD' - 30D'^2} xy \\
&= \frac{1}{36 - 6 - 30} e^{6x+y} + \frac{1}{D^2\left[\frac{D^2 - DD' - 30D'^2}{D^2}\right]} xy \\
&= \frac{1}{0} e^{6x+y} + \frac{1}{D^2\left[1 - \left(\frac{D'}{D} + \frac{30D'^2}{D^2}\right)\right]} xy \\
&= \frac{x}{2D - D'} e^{6x+y} + \frac{1}{D^2}\left[1 - \left(\frac{D'}{D} + \frac{30D'^2}{D^2}\right)\right]^{-1} xy \\
&= \frac{x}{12-1} e^{6x+y} + \frac{1}{D^2}\left[1 + \left(\frac{D'}{D} + \frac{30D'^2}{D^2}\right) + \left(\frac{D'}{D} + \frac{30D'^2}{D^2}\right)^2 + ...\right] xy \\
&= \frac{x}{11} e^{6x+y} + \frac{1}{D^2}\left[1 + \left(\frac{D'}{D}\right)\right] xy \text{ Since power of y is 1, D' enogh} \\
&= \frac{x}{11} e^{6x+y} + \frac{1}{D^2}\left[xy + \frac{x}{D}\right] \\
&= \frac{x}{11} e^{6x+y} + \frac{1}{D^2}\left[xy + \frac{x^2}{2}\right] \\
&= \frac{x}{11} e^{6x+y} + \frac{1}{D}\left[y.\frac{x^2}{2} + \frac{x^3}{6}\right] \\
&= \frac{x}{11} e^{6x+y} + y.\frac{x^3}{6} + \frac{x^4}{24} \\
\therefore z &= C.F + P.I \\
z &= f_1(y+6x) + f_2(y-5x) + \frac{x}{11} e^{6x+y} + \frac{x^3 y}{6} + \frac{x^4}{24}
\end{aligned}
$$

13. Solve $(D^2 - 4DD' + 4D'^2)z = xy + e^{2x+y}$

Solution:

A.E : Put $D = m$ and $D' = 1$

i.e $m^2 - 4m + 4 = 0$

$(m-2)(m-2) = 0$

$\Rightarrow m = 2, 2$

$\therefore$ C.F $= f_1(y+2x) + xf_2(y+2x)$

To find P.I

$$
\begin{aligned}
P.I &= \frac{1}{D^2 - 4DD' + 4D'^2} e^{2x+y} + \frac{1}{D^2 - 4DD' + 4D'^2} xy \\
&= \frac{1}{4 - 8 + 4} e^{2x+y} + \frac{1}{D^2\left[\frac{D^2 - 4DD' + 4D'^2}{D^2}\right]} xy
\end{aligned}
$$

| |
|---|
| $D = a = 2$ |
| $D' = b = 1$ |

$$\begin{aligned}
&= \frac{1}{0}e^{2x+y} + \frac{1}{D^2\left[1-\left(\frac{4D'}{D}-\frac{4D'^2}{D^2}\right)\right]}xy \\
&= \frac{x}{2D-4D'}e^{2x+y} + \frac{1}{D^2}\left[1-\left(\frac{4D'}{D}-\frac{4D'^2}{D^2}\right)\right]^{-1}xy \\
&= \frac{x}{4-4}e^{2x+y} + \frac{1}{D^2}\left[1+\left(\frac{4D'}{D}-\frac{4D'^2}{D^2}\right)+\left(\frac{4D'}{D}-\frac{4D'^2}{D^2}\right)^2+...\right]xy \\
&= \frac{x}{0}e^{2x+y} + \frac{1}{D^2}\left[1+\left(\frac{4D'}{D}\right)\right]xy \qquad \text{Since power of y is 1, D' enogh} \\
&= \frac{x^2}{2}e^{2x+y} + \frac{1}{D^2}\left[xy+\frac{4x}{D}\right] \\
&= \frac{x^2}{2}e^{2x+y} + \frac{1}{D^2}\left[xy+\frac{4x^2}{2}\right] \\
&= \frac{x^2}{2}e^{2x+y} + \frac{1}{D}\left[y.\frac{x^2}{2}+\frac{4x^3}{6}\right] \\
&= \frac{x^2}{2}e^{2x+y} + y.\frac{x^3}{6}+\frac{4x^4}{24} \\
&= \frac{x^2}{2}e^{2x+y} + \frac{x^3y}{6}+\frac{x^4}{6} \\
\therefore z &= C.F + P.I \\
z &= f_1(y+2x) + xf_2(y+2x) + \frac{x^2}{2}e^{2x+y} + \frac{x^3y}{6}+\frac{x^4}{6}
\end{aligned}$$

14. Solve $(D^2 + DD' - 6D'^2)z = x^2y + e^{3x+y}$

    Ans: $z = f_1(y+2x) + f_2(y-3x) + \frac{1}{6}e^{3x+y} + \frac{x^4y}{12} - \frac{x^5}{60}$

15. Solve $(D^3 - 2D^2D')z = 2e^{2x} + 3x^2y$

    Ans: $z = f_1(y) + xf_2(y) + f_3(y+2x) + \frac{1}{4}e^{2x} + \frac{x^5y}{60} - \frac{x^6}{60}$

16. Solve $(D^2 - 2DD')z = e^{2x-y} + x^3y$

    Ans: $z = f_1(y) + f_2(y+2x) + \frac{1}{8}e^{2x-y} + \frac{x^5y}{20} + \frac{x^6}{60}$

17. Solve $(D^2 + 3DD' + 2D'^2)z = \sin(2x+y) + x + y$

    Solution:

    A.E : Put $D = m$ and $D' = 1$

    i.e $m^2 + 3m + 2 = 0$

    $(m+1)(m+2) = 0$

    $\Rightarrow m = -1, -2$

    $\therefore$ C.F $= f_1(y-x) + f_2(y-2x)$

    To find P.I

$$\begin{aligned}
P.I &= \frac{1}{D^2+3DD'+2D'^2}\sin(2x+y) + \frac{1}{D^2+3DD'+2D'^2}(x+y) \\
&= \frac{1}{-4-6-2}\sin(2x+y) + \frac{1}{D^2\left[\frac{D^2+3DD'+2D'^2}{D^2}\right]}(x+y)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{-12}\sin(2x+y) + \frac{1}{D^2\left[1+\left(\frac{3D'}{D}+\frac{2D'^2}{D^2}\right)\right]}(x+y) \\
&= -\frac{1}{12}\sin(2x+y) + \frac{1}{D^2}\left[1+\left(\frac{3D'}{D}+\frac{2D'^2}{D^2}\right)\right]^{-1}(x+y) \\
&= -\frac{1}{12}\sin(2x+y) + \frac{1}{D^2}\left[1-\left(\frac{3D'}{D}+\frac{2D'^2}{D^2}\right)+\left(\frac{3D'}{D}+\frac{2D'^2}{D^2}\right)^2-...\right](x+y) \\
&= -\frac{1}{12}\sin(2x+y) + \frac{1}{D^2}\left[1-\left(\frac{3D'}{D}\right)\right](x+y) \\
&= -\frac{1}{12}\sin(2x+y) + \frac{1}{D^2}\left[x+y-\frac{3}{D}\right] \\
&= -\frac{1}{12}\sin(2x+y) + \frac{1}{D^2}[x+y-3x] \\
&= -\frac{1}{12}\sin(2x+y) + \frac{1}{D^2}[y-2x] \\
&= -\frac{1}{12}\sin(2x+y) + \frac{1}{D}\left[yx-\frac{2x^2}{2}\right] \\
&= -\frac{1}{12}\sin(2x+y) + y.\frac{x^2}{2}-\frac{2x^3}{6} \\
&= -\frac{1}{12}\sin(2x+y) + \frac{x^2y}{2}-\frac{x^3}{3} \\
\therefore z &= C.F + P.I \\
z &= f_1(y-x)+f_2(y-2x)-\frac{1}{12}\sin(2x+y)+\frac{x^2y}{2}-\frac{x^3}{3}
\end{aligned}
$$

| |
|---|
| $D^2 = -a^2 = -4$ |
| $D'^2 = -b^2 = -1$ |
| $D'^2 = -b^2 = -2$ |

18. Solve $(D^2+3DD'+2D'^2)z = x+y$

    Ans: $z = f_1(y-x)+f_2(y-2x)+\frac{x^2y}{2}-\frac{x^3}{3}$

19. Solve $(D^2-DD'-2D'^2)z = 2x+3y+e^{3x+4y}$

    Ans: $z = f_1(y+2x)+f_2(y-x)-\frac{1}{35}e^{3x+4y}+\frac{5x^3}{6}+\frac{3x^2y}{2}$

20. Solve $(D^2-6DD'+5D'^2)z = e^x\sin hy+xy$

    i.e, $(D^2-6DD'+5D'^2)z = e^x\left(\frac{e^y-e^{-y}}{2}\right)+xy$

    $(D^2-6DD'+5D'^2)z = \frac{1}{2}\left(e^{x+y}-e^{x-y}\right)+xy$

    $(D^2-6DD'+5D'^2)z = \frac{1}{2}e^{x+y}-\frac{1}{2}e^{x-y}+xy$

    Solution:

    A.E : Put $D = m$ and $D' = 1$

    i.e $m^2-6m+5=0$

    $(m-1)(m-5)=0$

    $\Rightarrow m = 1, 5$

    $\therefore$ C.F $= f_1(y+x)+f_2(y+5x)$

    To find P.I

$$
P.I = \frac{1}{D^2-6DD'+5D'^2}\frac{1}{2}e^{x+y}+\frac{1}{D^2-6DD'+5D'^2}\frac{1}{2}e^{x-y}+\frac{1}{D^2-6DD'+5D'^2}xy
$$

$$\begin{aligned}
&= \frac{1}{2}\left[\frac{1}{1-6+5}e^{x+y}\right]+\frac{1}{2}\left[\frac{1}{1+6+5}\right]e^{x-y}+\frac{1}{D^2\left[\frac{D^2-6DD'+5D'^2}{D^2}\right]}xy \\
&= \frac{1}{2}\left[\frac{1}{0}\right]e^{x+y}+\frac{1}{2}\left[\frac{1}{12}\right]e^{x-y}+\frac{1}{D^2\left[1-\left(\frac{6D'}{D}-\frac{5D'^2}{D^2}\right)\right]}xy \\
&= \frac{1}{2}\left[\frac{x}{2D-6D'}\right]e^{x+y}+\frac{1}{24}e^{x-y}+\frac{1}{D^2}\left[1-\left(\frac{6D'}{D}-\frac{5D'^2}{D^2}\right)\right]^{-1}xy \\
&= \frac{1}{2}\left[\frac{x}{2-6}\right]e^{x+y}+\frac{1}{24}e^{x-y}+\frac{1}{D^2}\left[1+\left(\frac{6D'}{D}-\frac{5D'^2}{D^2}\right)+\left(\frac{6D'}{D}-\frac{5D'^2}{D^2}\right)^2+...\right]xy \\
&= \frac{1}{2}\left[\frac{x}{-4}\right]e^{x+y}+\frac{1}{24}e^{x-y}+\frac{1}{D^2}\left[1+\left(\frac{6D'}{D}\right)\right]xy \\
&= -\frac{1}{8}e^{x+y}+\frac{1}{24}e^{x-y}+\frac{1}{D^2}\left[xy+\frac{6x}{D}\right] \\
&= -\frac{1}{8}e^{x+y}+\frac{1}{24}e^{x-y}+\frac{1}{D^2}\left[xy+\frac{6x^2}{2}\right] \\
&= -\frac{1}{8}e^{x+y}+\frac{1}{24}e^{x-y}+\frac{1}{D}\left[y\frac{x^2}{2}+\frac{6x^3}{6}\right] \\
&= -\frac{1}{8}e^{x+y}+\frac{1}{24}e^{x-y}+y\frac{x^3}{6}+\frac{6x^4}{24} \\
\therefore z &= C.F + P.I \\
z &= f_1(y+x)+f_2(y+5x)-\frac{1}{8}e^{x+y}+\frac{1}{24}e^{x-y}+y\frac{x^3}{6}+\frac{x^4}{4}
\end{aligned}$$

21. Solve $(D^2+2DD'+D'^2)z = \sin h(x+y)+e^{x+2y}$

Ans: $z = f_1(y-x)+xf_2(y-x)+\frac{1}{8}[e^{x+y}-e^{-(x+y)}]+\frac{1}{9}e^{x+2y}$ (OR)

Ans: $z = f_1(y-x)+xf_2(y-x)+\frac{1}{4}\sin h(x+y)+\frac{1}{9}e^{x+2y}$

22. Solve $(D^3+D^2D'-DD'^2-D'^3)z = \sin 2x\cos y$

Solution:

$(D^3+D^2D'-DD'^2-D'^3)z = \frac{1}{2}[\sin(2x+y)+\sin(2x-y)]$

$(D^3+D^2D'-DD'^2-D'^3)z = \frac{1}{2}\sin(2x+y)+\frac{1}{2}\sin(2x-y)$

A.E : Put $D = m$ and $D' = 1$

i.e $m^3+m^2-m-1=0$

| | | | | |
|---|---|---|---|---|
| | 1 | 1 | −1 | −1 |
| 1 | | 1 | 2 | 1 |
| | 1 | 2 | 1 | 0 |

i.e $m^2+2m+1=0$

$(m+1)(m+1)=0$

$\Rightarrow m = -1, -1$

$\therefore m = 1, -1, -1$

C.F $= f_1(y+x)+f_2(y-x)+xf_3(y-x)$

To find P.I

| |
|---|
| $D^2 = -a^2 = -4$ |
| $D'^2 = -b^2 = -1$ |
| $D'^2 = -b^2 = -2$ |

$$\begin{aligned}
P.I &= \frac{1}{D^3+D^2D'-DD'^2-D'^3}\frac{1}{2}\sin(2x+y)+\frac{1}{D^3+D^2D'-DD'^2-D'^3}\frac{1}{2}\sin(2x-y)\\
&= \frac{1}{2}\left[\frac{1}{-4D-4D'+D+D'}\right]\sin(2x+y)+\frac{1}{2}\left[\frac{1}{-4D-4D'+D+D'}\right]\sin(2x-y)\\
&= \frac{1}{2}\left[\frac{1}{-3D-3D'}\right]\sin(2x+y)+\frac{1}{2}\left[\frac{1}{-3D-3D'}\right]\sin(2x-y)\\
&= \frac{1}{2}\left[\frac{D}{-3D^2-3DD'}\right]\sin(2x+y)+\frac{1}{2}\left[\frac{D}{-3D^2-3DD'}\right]\sin(2x-y)\\
&= \frac{1}{2}\left[\frac{\cos(2x+y)2}{-3(-4)-3(-2)}\right]+\frac{1}{2}\left[\frac{\cos(2x-y)(2)}{-3(-4)-3(2)}\right]\\
&= \frac{\cos(2x+y)}{18}+\frac{\cos(2x-y)}{6}\\
\therefore z &= C.F + P.I\\
z &= f_1(y+x)+f_2(y-x)+xf_3(y-x)+\frac{\cos(2x+y)}{18}+\frac{\cos(2x-y)}{6}
\end{aligned}$$

23. Solve $(D^3+D^2D'-DD'^2-D'^3)z = e^x\cos 2y$

Solution:

A.E : Put $D = m$ and $D' = 1$

i.e $m^3+m^2-m-1=0$

| | | | | |
|---|---|---|---|---|
| | 1 | 1 | −1 | −1 |
| 1 | | 1 | 2 | 1 |
| | 1 | 2 | 1 | 0 |

i.e $m^2+2m+1=0$

$(m+1)(m+1)=0$

$\Rightarrow m = -1, -1$

$\therefore m = 1, -1, -1$

C.F $= f_1(y+x)+f_2(y-x)+xf_3(y-x)$

To find P.I

| |
|---|
| $D = D + a = D + 1$ |
| $D' = D' + b = D'$ |

$$\begin{aligned}
P.I &= \frac{1}{D^3+D^2D'-DD'^2-D'^3}e^x\cos 2y\\
&= e^x\frac{1}{(D+1)^3+(D+1)^2D'-(D+1)D'^2-D'^3}\cos 2y\\
&= e^x\frac{\text{R.P of } e^{i2y}}{(D+1)^3+(D+1)^2D'-(D+1)D'^2-D'^3}\\
&= e^x\frac{\text{R.P of } e^{i2y}}{(0+1)^3+(0+1)^2(2i)-(0+1)(2i)^2-(2i)^3}\\
&= e^x\frac{\text{R.P of } e^{i2y}}{1+2i+4+8i}\\
&= e^x\frac{\text{R.P of } e^{i2y}}{5+10i}\\
&= e^x\frac{\text{R.P of } e^{i2y}}{5(1+2i)}\\
&= \frac{e^x}{5}\text{R.P of }\frac{(1-2i)(\cos 2y+i\sin 2y)}{(1+2i)(1-2i)}
\end{aligned}$$

$$= \frac{e^x}{5}\frac{\cos 2y + 2\sin 2y}{1+4}$$
$$= \frac{e^x}{25}(\cos 2y + 2\sin 2y)$$
$$\therefore z = C.F + P.I$$
$$z = f_1(y+x) + f_2(y-x) + xf_3(y-x) + \frac{e^x}{25}(\cos 2y + 2\sin 2y)$$

24. $r + s - 6t = y\cos x$

Solution:

Given : $(D^2 + DD' - 6D'^2) = y\cos x$ A.E : Put $D = m$ and $D' = 1$

i.e $m^2 + m - 6 = 0$

$(m+3)(m-2) = 0$

$\Rightarrow m = -3, 2$

C.F $= f_1(y-3x) + f_2(y+2x)$

To find P.I

$$P.I = \frac{1}{D^2 + DD' - 6D'^2} y\cos x$$
$$= \frac{1}{(D+3D')(D-2D')} y\cos x$$
$$= \frac{1}{(D+3D')}\int (c-2x)\cos x dx \text{ where } y = c - 2x$$
$$= \frac{1}{(D+3D')}[(c-2x)\sin x - (-2)(-\cos x)]$$
$$= \frac{1}{(D+3D')}[y\sin x - 2\cos x)]$$
$$= \int [(c+3x)\sin x - 2\cos x)]\, dx \text{ where } y = c + 3x$$
$$= [(c+3x)(-\cos x) - 3(-\sin x) - 2\sin x]$$
$$= [y(-\cos x) + 3\sin x) - 2\sin x]$$
$$= [-y\cos x) + \sin x]$$
$$\therefore z = C.F + P.I$$
$$z = f_1(y-3x) + f_2(y+2x) - y\cos x) + \sin x$$

25. $(D^2 + 2DD' + D'^2) = -x\sin y$

Solution:

A.E : Put $D = m$ and $D' = 1$

i.e $m^2 + 2m + 1 = 0$

$(m+1)(m+1) = 0$

$\Rightarrow m = -1, -1$

C.F $= f_1(y-x) + xf_2(y-x)$

To find P.I

$$\begin{aligned}
P.I &= \frac{1}{D^2+2DD'+D'^2} - x\sin y \\
&= -\frac{1}{(D+D')(D+D')} x\sin y \\
&= -\frac{1}{(D+D')}\int x\sin(c+x)dx \text{ where } y=c+x \\
&= -\frac{1}{(D+D')}\left[x(-\cos(c+x)-(1)(-\sin(c+x))\right] \\
&= -\frac{1}{(D+D')}\left[-x\cos y+\sin y)\right] \\
&= -\int\left[-x\cos(c+x)+\sin(c+x)\right]dx \text{ where } y=c+x \\
&= -\left[(-x)(\sin(c+x))-(-1)(-\cos(c+x))+(-\cos(c+x))\right] \\
&= -\left[-x\sin y)-\cos y)-\cos y\right] \\
&= -\left[-x\sin y)-2\cos y\right] \\
&= x\sin y+2\cos y \\
\therefore z &= C.F+P.I \\
z &= f_1(y-x)+xf_2(y-x)+x\sin y+2\cos y
\end{aligned}$$

26. Solve $\dfrac{\partial^2 z}{\partial x^2}-\dfrac{\partial^2 z}{\partial x\partial y}-2\dfrac{\partial^2 z}{\partial y^2}=(y-1)e^x$

Solution:

Given: $(D^2-DD'-2D'^2)=(y-1)e^x$

A.E : Put $D=m$ and $D'=1$

i.e $m^2-m-2=0$

$(m+1)(m-2)=0$

$\Rightarrow m=-1,2$

C.F $=f_1(y-x)+f_2(y+2x)$

To find P.I

$$\begin{aligned}
P.I &= \frac{1}{D^2-DD'-2D'^2}(y-1)e^x \\
&= d\frac{1}{(D+D')(D-2D')}(y-1)e^x \\
&= \frac{1}{(D+D')}\int(c-2x-1)e^x dx \text{ where } y=c-2x \\
&= \frac{1}{(D+D')}\left[(c-2x-1)e^x-(-2)(e^x)\right] \\
&= \frac{1}{(D+D')}\left[(y-1)e^x+2e^x\right] \\
&= \int\left[(c+x-1)e^x+2e^x\right]dx \text{ where } y=c+x \\
&= \left[(c+x-1)e^x-(1)e^x+2e^x\right] \\
&= (y-1)(e^x)+e^x
\end{aligned}$$

$$\begin{aligned} &= ye^x - e^x + e^x \\ &= ye^x \\ \therefore z &= C.F + P.I \\ z &= f_1(y-x) + f_2(y+2x) + ye^x \end{aligned}$$

27. Solve $(D^2 - 5DD' + 6D'^2)z = y\sin x$

 Ans: $z = f_1(y+2x) + f_2(y+3x) + 5\cos x - y\sin x$

28. Solve $(D^2 - 2DD' + D'^2)z = x^2y^2e^{x+y}$

 Ans: $z = f_1(y+x) + xf_2(y+x) + \left[\frac{x^4y^2}{12} + \frac{x^5y}{15} + \frac{x^6}{60}\right]e^{x+y}$

(Non-Homogeneous)Form:

| Form | C.F |
|---|---|
| $(D - m_1D' - c_1)(D - m_2D' - c_2)z = F(x,y)$ | $e^{c_1x}f_1(y+m_1x) + e^{c_2x}f_2(y+m_2x)$ |
| $(D - mD' - c)^2z = F(x,y)$ | $e^{cx}f_1(y+mx) + xe^{cx}f_2(y+mx)$ |
| $(D' - m_1D - c_1)(D' - m_2D - c_2)z = F(x,y)$ | $e^{c_1y}f_1(x+m_1y) + e^{c_2y}f_2(x+m_2y)$ |
| $(D' - mD - c)^2z = F(x,y)$ | $e^{cy}f_1(x+my) + ye^{cy}f_2(x+my)$ |

To find P.I

P.I is same as Homogeneous

29. $(D^2 - 2DD' + D'^2 - 3D + 3D' + 2)z = e^{2x-y}$

 Solution:

$$\begin{aligned} \left[(D-D')^2 - 1 - 3D + 3D' + 3\right]z &= e^{2x-y} \\ \left[(D-D'-1)(D-D'+1) - 3(D-D'-1)\right]z &= e^{2x-y} \\ (D-D'-1)\left[(D-D'+1) - 3\right]z &= e^{2x-y} \\ (D-D'-1)(D-D'-2)z &= e^{2x-y} \\ \text{Here } m_1 = 1, c_1 &= 1 \\ m_2 = 1, c_2 &= 2 \\ C.F = e^xf_1(y+x) &+ e^xf_2(y+2x) \end{aligned}$$

To find P.I

$D = a = 2$

$D' = b = -1$

$$\begin{aligned} P.I &= \frac{1}{D^2 - 2DD' + D'^2 - 3D + 3D' + 2}e^{2x-y} \\ &= \frac{1}{4+4+1-6-3+2}e^{2x-y} \\ &= \frac{1}{2}e^{2x-y} \\ \therefore z &= C.F + P.I \\ z &= e^xf_1(y+x) + e^xf_2(y+2x) + \frac{1}{2}e^{2x-y} \end{aligned}$$

30. $(D^2 + 2DD' + D'^2 - 2D - 2D')z = \sin(x + 2y)$

Solution:

$$\left[(D + D')^2 - 2(D + D')\right] z = \sin(x + 2y)$$
$$(D + D')\left[(D + D') - 2\right] z = \sin(x + 2y)$$
$$(D - (-1)D')(D - (-1)D' - 2)z = \sin(x + 2y)$$

Here $m_1 = -1, c_1 = 0$

$m_2 = -1, c_2 = 2$

$C.F = f_1(y - x) + e^{2x} f_2(y - x)$

To find P.I

| |
|---|
| $D^2 = -a^2 = -1$ |
| $D'^2 = -b^2 = -4$ |
| $DD' = -ab = -2$ |

$$\begin{aligned}
P.I &= \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'} \sin(x + 2y) \\
&= \frac{1}{-1 - 4 - 4 - 2D - 2D'} \sin(x + 2y) \\
&= \frac{D}{-9D - 2D^2 - 2DD'} \sin(x + 2y) \\
&= \frac{\cos(x + 2y)}{-9D + 2 + 4} \\
&= \frac{(-9D - 6)\cos(x + 2y)}{(-9D + 6)(-9D - 6)} \\
&= \frac{(-9(-\sin(x + 2y)) - 6\cos(x + 2y)}{81D^2 - 36} \\
&= \frac{9\sin(x + 2y) - 6\cos(x + 2y)}{-81 - 36} \\
&= \frac{9\sin(x + 2y) - 6\cos(x + 2y)}{-117} \\
&= -\frac{3}{39}\sin(x + 2y) + \frac{2}{39}\cos(x + 2y) \\
\therefore z &= C.F + P.I \\
z &= f_1(y - x) + e^{2x} f_2(y - x) - \frac{3}{39}\sin(x + 2y) + \frac{2}{39}\cos(x + 2y)
\end{aligned}$$

31. $(D^2 - 2DD' + D'^2 - 3D + 3D' + 2)z = e^{2x-y}$

Solution:

$$\left[(D - D')^2 - 1 - 3D + 3D' + 3\right] z = e^{2x-y}$$
$$[(D - D' - 1)(D - D' + 1) - 3(D - D' - 1)] z = e^{2x-y}$$
$$(D - D' - 1)\left[(D - D' + 1) - 3\right] z = e^{2x-y}$$
$$(D - D' - 1)(D - D' - 2)z = e^{2x-y}$$

Here $m_1 = 1, c_1 = 1$

$m_2 = 1, c_2 = 2$

$C.F = e^x f_1(y + x) + e^x f_2(y + 2x)$

To find P.I

| |
|---|
| $D = a = 2$ |
| $D' = b = -1$ |

$$\begin{aligned}
P.I &= \frac{1}{D^2 - 2DD' + D'^2 - 3D + 3D' + 2} e^{2x-y} \\
&= \frac{1}{4 + 4 + 1 - 6 - 3 + 2} e^{2x-y}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}e^{2x-y} \\
\therefore z &= C.F + P.I \\
z &= e^{x} f_1(y+x) + e^{x} f_2(y+2x) + \frac{1}{2}e^{2x-y}
\end{aligned}
$$

32. $(2D^2 - DD' - D'^2 + 6D + 3D')z = xe^y$

Solution:

$$
\begin{aligned}
&\left[2D^2 - 2DD' + DD' - D'^2 + 6D + 3D'\right] z = xe^y \\
&\left[2D^2 - 2DD' + 6D + DD' - D'^2 + 3D'\right] z = xe^y \\
&[2D(D - D' + 3) + D'(D - D' + 3)]\, z = xe^y \\
&[(D - D' + 3)(2D + D')]\, z = xe^y
\end{aligned}
$$

C.F $\Rightarrow [(D - D' + 3)(2D + D')]\, z = 0$

$\left[(D - D' + 3)(D + \frac{1}{2}D')\right] z = 0$

Here $m_1 = 1, c_1 = -3$

$m_2 = \frac{-1}{2}, c_2 = 0$

$C.F = e^{-3x} f_1(y+x) + e^{0x} f_2\left(y - \frac{1}{2}x\right)$

To find P.I

| $D = D + a = D$ |
|---|
| $D' = D' + b = D' + 1$ |

$$
\begin{aligned}
P.I &= \frac{1}{2D^2 - DD' - D'^2 + 6D + 3D'} xe^y \\
&= e^y \frac{1}{2D^2 - D(D'+1) - (D'+1)^2 + 6D + 3(D'+1)} x \\
&= e^y \frac{D}{2D^2 - DD' - D - D'^2 - 2D' - 1 + 6D + 3D' + 3} x \\
&= e^y \frac{1}{2D^2 - DD' + 5D - D'^2 - 2D' + D' + 2} x \\
&= e^y \frac{1}{2\left[1 + \left(\frac{2D^2 - DD' + 5D - D'^2 - 2D' + D' + 2}{2}\right)\right]} x \\
&= \frac{e^y}{2}\left[1 + \left(\frac{2D^2 - DD' + 5D - D'^2 - 2D' + D' + 2}{2}\right)\right]^{-1} x \\
&= \frac{e^y}{2}\left[1 - \left(\frac{2D^2 - DD' + 5D - D'^2 - 2D' + D' + 2}{2}\right) + \left(\frac{2D^2 - DD' + 5D - D'^2 - 2D' + D' + 2}{2}\right)^2\right] x \\
&= \frac{e^y}{2}\left[1 - \frac{5}{2}D\right] x \\
&= \frac{e^y}{2}\left[x - \frac{5}{2}\right] \\
\therefore z &= C.F + P.I \\
z &= e^{-3x} f_1(y+x) + e^{0x} f_2\left(y - \frac{1}{2}x\right) + \frac{e^y}{2}\left[x - \frac{5}{2}\right]
\end{aligned}
$$

33. $(D^2 - 3DD' + 2D'^2 + 2D - 2D')z = x + y + \sin(2x+y)$

Ans: $z = f_1(y+x) + e^{2x} f_2(y+2x) - \frac{1}{2}(x^2 + xy) - \frac{1}{2}\cos(2x+y)$

## Lagrange's Method

Form: $Pp + Qq = R$

Auxiliary Equation:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

34. Solve $x(y-z)p + y(z-x)q = z(x-y)$

Solution:

Lagranges's type: $Pp + Qq = R$

Here $P = x(y-z), Q = y(z-x), R = z(x-y)$

Auxiliary Equation:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

Use Lagrangian Multiplier 1,1,1

$$\begin{aligned} \frac{dx+dy+dz}{x(y-z)+y(z-x)+z(x-y)} &= \frac{dx+dy+dz}{0} \\ \Rightarrow dx+dy+dz &= 0 \\ \text{Integrating} \\ \Rightarrow x+y+z &= c_1 \end{aligned}$$

Use Lagrangian Multiplier $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$

$$\begin{aligned} \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y-z+z-x+x-y} &= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0} \\ \Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz &= 0 \\ \text{Integrating} \\ \Rightarrow \log x + \log y + \log z &= \log c_2 \\ \log xyz &= \log c_2 \\ \Rightarrow xyz &= c_2 \end{aligned}$$

$\therefore$ General Solution: $\phi(x+y+z, xyz) = 0$

35. Solve $x(y^2-z^2)p + y(z^2-x^2)q = z(x^2-y^2)$

Solution:

Lagranges's type: $Pp + Qq = R$

Here $P = x(y^2-z^2), Q = y(z^2-x^2), R = z(x^2-y^2)$

Auxiliary Equation:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)}$$

Use Lagrangian Multiplier x,y,z

$$\frac{xdx+ydy+zdz}{x^2(y^2-z^2)+y^2(z^2-x^2)+z^2(x^2-y^2)} = \frac{xdx+ydy+zdz}{0}$$
$$\Rightarrow xdx+ydy+zdz = 0$$

Integrating

$$\Rightarrow \frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2} = \frac{c_1}{2}$$
$$x^2+y^2+z^2 = c_1$$

Use Lagrangian Multiplier $\frac{1}{x},\frac{1}{y},\frac{1}{z}$

$$\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{y^2-z^2+z^2-x^2+x^2-y^2} = \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$$
$$\Rightarrow \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz = 0$$

Integrating

$$\Rightarrow \log x+\log y+\log z = \log c_2$$
$$\log xyz = \log c_2$$
$$\Rightarrow xyz = c_2$$

$\therefore$ General Solution: $\phi(x^2+y^2+z^2, xyz) = 0$

36. Solve $x^2(y-z)p+y^2(z-x)q = z^2(x-y)$

Solution:

Lagranges's type: $Pp+Qq=R$

Here $P = x^2(y-z), Q = y^2(z-x), R = z^2(x-y)$

Auxiliary Equation:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$
$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

Use Lagrangian Multiplier $\frac{1}{x^2},\frac{1}{y^2},\frac{1}{z^2}$

$$\frac{\frac{1}{x^2}dx+\frac{1}{y^2}dy+\frac{1}{z^2}dz}{y-z+z-x+x-y} = \frac{\frac{1}{x^2}dx+\frac{1}{y^2}dy+\frac{1}{z^2}dz}{0}$$
$$\Rightarrow \frac{1}{x^2}dx+\frac{1}{y^2}dy+\frac{1}{z^2}dz = 0$$

Integrating

$$\Rightarrow \frac{-1}{x}-\frac{1}{y}-\frac{1}{z} = -c_1$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = c_1$$

Use Lagrangian Multiplier $\frac{1}{x},\frac{1}{y},\frac{1}{z}$

$$\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{x(y-z)+y(z-x)+z(x-y)} = \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$$

$$\Rightarrow \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz = 0$$

Integrating

$$\Rightarrow \log x+\log y+\log z = \log c_2$$

$$\log xyz = \log c_2$$

$$\Rightarrow xyz = c_2$$

$\therefore$ General Solution: $\phi\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}, xyz\right) = 0$

37. Solve $(mz-ny)p+(nx-lz)q = (ly-mx)$

Solution:

Lagranges's type: $Pp+Qq = R$

Here $P = mz-ny, Q = nx-lz, R = ly-mx$

Auxiliary Equation:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$$

Use Lagrangian Multiplier x,y,z

$$\frac{xdx+ydy+zdz}{x(mz-ny)+y(nx-lz)+z(ly-mx)} = \frac{xdx+ydy+zdz}{0}$$

$$\Rightarrow xdx+ydy+zdz = 0$$

Integrating

$$\Rightarrow \frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2} = \frac{c_1}{2}$$

$$x^2+y^2+z^2 = c_1$$

Use Lagrangian Multiplier $l, m, n$

$$\frac{ldx+mdy+ndz}{l(mz-ny)+m(nx-lz)+n(ly-mx)} = \frac{ldx+mdy+ndz}{0}$$

$$\Rightarrow ldx+mdy+ndz = 0$$

Integrating

$$\Rightarrow lx+my+nz = c_2$$

$\therefore$ General Solution: $\phi(x^2+y^2+z^2, lx+my+nz)=0$

38. Solve $(3z-4y)p+(4x-2z)q=(2y-3x)$

    Refer previous question here $l=2, m=3, n=4$

39. Solve $(y-xz)p+(yz-x)q=(x+y)(x-y)$

    Solution:

    Lagranges's type: $Pp+Qq=R$

    Here $P=y-xz, Q=yz-x, R=x^2-y^2$

    Auxiliary Equation:
    $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$
    $\frac{dx}{y-xz}=\frac{dy}{yz-x}=\frac{dz}{x^2-y^2}$
    Use Lagrangian Multiplier x,y,z

$$\begin{aligned}\frac{xdx+ydy+zdz}{x(y-xz)+y(yz-x)+z(x^2-y^2)} &= \frac{xdx+ydy+zdz}{0}\\ \Rightarrow xdx+ydy+zdz &= 0\\ \text{Integrating}\\ \Rightarrow \frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2} &= \frac{c_1}{2}\\ x^2+y^2+z^2 &= c_1\end{aligned}$$

Use Lagrangian Multiplier $y,x,1$

$$\begin{aligned}\frac{ydx+xdy+dz}{y(y-xz)+x(yz-x)+(x^2-y^2)} &= \frac{ydx+xdy+dz}{0}\\ \Rightarrow ydx+xdy+dz &= 0\\ d(xy)+dz &= 0\\ \text{Integrating}\\ \Rightarrow xy+z &= c_2\end{aligned}$$

$\therefore$ General Solution: $\phi(x^2+y^2+z^2, xy+z)=0$

40. Solve $x(y^2+z)p-y(x^2+z)q=z(x^2-y^2)$

    Hint: Multipliers, $\frac{1}{x},\frac{1}{y},\frac{1}{z}$ and $x,y,-1$

    Ans: $\phi(x^2+y^2-2z, xyz)=0$

41. Solve $(y^2+z^2)p-xyq+xz=0$

    Solution:

    Lagranges's type: $Pp+Qq=R$

    Here $P=y^2+z^2, Q=-xy, R=-xz$

    Auxiliary Equation:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$
$$\frac{dx}{y^2+z^2} = \frac{dy}{-xy} = \frac{dz}{-xz}$$

Taking 2nd and 3rd member,

$$\frac{dy}{-xy} = \frac{dz}{-xz}$$

$$\begin{aligned} \frac{dy}{y} &= \frac{dz}{z} \\ \frac{dy}{y} - \frac{dz}{z} &= 0 \\ \log y - \log z &= \log c_1 \\ \frac{y}{z} &= c_1 \end{aligned}$$

Use Lagrangian Multiplier x,y,z

$$\begin{aligned} \frac{xdx+ydy+zdz}{x(y^2+z^2)+y(-xy)+z(-xz)} &= \frac{xdx+ydy+zdz}{0} \\ \Rightarrow xdx+ydy+zdz &= 0 \\ \text{Integrating} & \\ \Rightarrow \frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2} &= \frac{c_2}{2} \\ x^2+y^2+z^2 &= c_2 \end{aligned}$$

$\therefore$ General Solution: $\phi\left(\frac{y}{z}, x^2+y^2+z^2\right) = 0$

42. Solve $(x^2-y^2-z^2)p + 2xyq - 2xz = 0$

Solution:

Lagranges's type: $Pp + Qq = R$

Here $P = x^2-y^2-z^2, Q = 2xy, R = 2xz$

Auxiliary Equation:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$
$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$$

Taking 2nd and 3rd member,

$$\begin{aligned} \frac{dy}{2xy} &= \frac{dz}{2xz} \\ \frac{dy}{y} &= \frac{dz}{z} \\ \frac{dy}{y} - \frac{dz}{z} &= 0 \\ \log y - \log z &= \log c_1 \\ \frac{y}{z} &= c_1 \end{aligned}$$

Use Lagrangian Multiplier x,y,z

$$\begin{aligned}
\frac{xdx+ydy+zdz}{x(x^2-y^2-z^2)+y(2xy)+z(2xz)} &= \frac{dy}{2xy} \\
\frac{xdx+ydy+zdz}{x(x^2-y^2-z^2+2y^2+2z^2)} &= \frac{dy}{2xy} \\
\frac{xdx+ydy+zdz}{x^2+y^2+z^2} &= \frac{dy}{2y} \\
\frac{2xdx+2ydy+2zdz}{x^2+y^2+z^2} &= \frac{dy}{y} \\
\frac{2xdx+2ydy+2zdz}{x^2+y^2+z^2}\frac{dy}{y} &= 0 \\
\text{Integrating} & \\
\Rightarrow \log(x^2+y^2+z^2)-\log y &= \log c_2 \\
\log\frac{x^2+y^2+z^2}{y} &= \log c_2 \\
\frac{x^2+y^2+z^2}{y} &= c_2
\end{aligned}$$

$\therefore$ General Solution: $\phi\left(\frac{y}{z},\frac{x^2+y^2+z^2}{y}\right)=0$

43. Solve $(x^2-yz)p+(y^2-zx)q=z^2-xy$

Solution:

Lagranges's type: $Pp+Qq=R$

Here $P=x^2-yz, Q=y^2-zx, R=z^2-xy$

Auxiliary Equation:

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$$\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy}$$

$$\frac{dx-dy}{x^2-yz-y^2+zx}=\frac{dy-dz}{y^2-zx-z^2+xy}=\frac{dx+dy+dz}{x^2-yz+y^2-zx+z^2-xy}=\frac{xdx+ydy+zdz}{x^3-xyz+y^3-xyz+z^3-xyz}$$

Taking 1nd and 2nd member,

$$\begin{aligned}
\frac{dx-dy}{x^2-y^2-yz+zx} &= \frac{dy-dz}{y^2-z^2-zx+xy} \\
\frac{d(x-y)}{(x+y)(x-y)+z(x-y)} &= \frac{d(y-z)}{(y+z)(y-z)+x(y-z)} \\
\frac{d(x-y)}{(x-y)[x+y+z]} &= \frac{d(y-z)}{(y-z)[y+z+x]} \\
\frac{d(x-y)}{x-y} &= \frac{d(y-z)}{y-z} \\
\frac{d(x-y)}{x-y}-\frac{d(y-z)}{y-z} &= 0 \\
\text{Integrating} & \\
\log(x-y)-\log(y-z) &= \log c_1 \\
\log\frac{x-y}{y-z} &= \log c_1 \\
\frac{x-y}{y-z} &= c_1
\end{aligned}$$

Taking 3rd and 4th member,

$$\begin{aligned}
\frac{dx+dy+dz}{x^2-yz+y^2-zx+z^2-xy} &= \frac{xdx+ydy+zdz}{x^3-xyz+y^3-xyz+z^3-xyz} \\
\frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx} &= \frac{xdx+ydy+zdz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} \\
dx+dy+dz &= \frac{xdx+ydy+zdz}{x+y+z} \\
(x+y+z)dx+dy+dz &= xdx+ydy+zdz \\
(x+y+z)dx+dy+dz-xdx-ydy-zdz &= 0 \\
\text{Integrating} \\
\frac{(x+y+z)^2}{2}-\frac{x^2}{2}-\frac{y^2}{2}-\frac{z^2}{2} &= \frac{c_2}{2} \\
(x+y+z)^2-x^2-y^2-z^2 &= c_2 \\
x^2+y^2+z^2+2xy+2yz+2xz-x^2-y^2-z^2 &= c_2 \\
2xy+2yz+2xz &= c_2 \\
xy+yz+xz &= c
\end{aligned}$$

$\therefore$ General Solution: $\phi\left(\frac{x-y}{y-z}, xy+yz+xz\right)=0$

44. Solve $(x^2+y^2+yz)p+y(x^2+y^2-zx)q=z(x+y)$
Ans: $\phi(x-y-z, \frac{x^2+y^2}{z^2})=0$

# Unit-2

# Fourier Series

### Condition for a Fourier Expansion:[Dirichlet's Conditions]

Any function $f(x)$ can be developed as a Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ where $a_0, a_n, b_n$ are constants, provided.

1. $f(x)$ is periodic, single-valued and finite.
2. $f(x)$ has a finite number of finite discontinuities in any one period and no infinite discontinuity.
3. $f(x)$ has at the most number of maxima and minima.

### Euler's Formula for the Fourier Coefficients

If a function $f(x)$ defined in $c < x < c+2\pi$ can be expanded as the infinite trigonometric series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\begin{aligned} \text{where } a_0 &= \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx \end{aligned}$$

**NOTE:1**

**Deduction:[Continuity and Discontinuity]**

Consider the function $f(x) = \begin{cases} f_1(x), & \text{for } a \le x < c; \\ f_2(x), & \text{for } c < x \le b. \end{cases}$

1. If $x = a$ is a point of continuity then use $f(x) = f(a)$
2. If $x = a$ is a point of discontinuity at the end point then use $f(x) = \frac{f(a) + f(b)}{2}$
3. If $x = c$ is a point of discontinuity at the mid point then use $f(x) = \frac{f(-c) + f(+c)}{2}$

**NOTE:2**

1. To change $\sum\limits_{n=odd}^{\infty}$ to $\sum\limits_{n=1}^{\infty}$ replace $n = 2n - 1$
2. To change $\sum\limits_{n=even}^{\infty}$ to $\sum\limits_{n=1}^{\infty}$ replace $n = 2n$
3. To change $\sum\limits_{n=odd}^{\infty}$ to $\sum\limits_{n=0}^{\infty}$ replace $n = 2n + 1$

**NOTE 3: PARSEVAL'S IDENTITY(higher power derivation)**

For Full Range: $\frac{1}{2\pi}\int\limits_0^{2\pi}[f(x)]^2dx = \frac{a_0^2}{4} + \frac{1}{2}\sum(a_n^2 + b_n^2)$

For Half Range Cosine: $\frac{1}{\pi}\int\limits_0^{\pi}[f(x)]^2dx = \frac{a_0^2}{4} + \frac{1}{2}\sum a_n^2$

For Full Range Sine: $\frac{1}{\pi}\int\limits_0^{\pi}[f(x)]^2dx = \frac{1}{2}\sum b_n^2$

## Full range $\pi$ Type

1. If $f(x) = \frac{1}{2}(\pi - x)$ find the Fourier series of the period $2\pi$ in the interval $(0, 2\pi)$.Hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$

   **Solution:**

   The Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx................(1)$$

**To find $a_0$**

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int\limits_0^{2\pi} f(x)dx \\ &= \frac{1}{\pi}\int\limits_0^{2\pi} \frac{1}{2}(\pi - x)dx \\ &= \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{2\pi} \\ &= \frac{1}{2\pi}[2\pi^2 - 2\pi^2 - \{0 - 0\}] \\ &= 0 \end{aligned}$$

**To find $a_n$**

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx \\ &= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\cos nx dx \\ &= \frac{1}{2\pi}\left[(\pi - x)\frac{\sin nx}{n} - (-1)\frac{(-\cos nx)}{n^2}\right]_0^{2\pi} \\ &= \frac{1}{2\pi}\left[(\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]_0^{2\pi} \\ &= \frac{1}{2\pi}\left[0 - \frac{1}{n^2} - \left\{0 - \frac{1}{n^2} - 0\right\}\right] \\ &= 0 \end{aligned}$$

**To find $b_n$**

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx \\ &= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\sin nx dx \\ &= \frac{1}{2\pi}\left[(\pi - x)\frac{(-\cos nx)}{n} - (-1)\frac{(-\sin nx)}{n^2}\right]_0^{2\pi} \\ &= \frac{1}{2\pi}\left[-(\pi - x)\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right]_0^{2\pi} \\ &= \frac{1}{2\pi}\left[-(-\pi)\frac{1}{n} - 0 - \left\{-\pi\frac{1}{n} - 0\right\}\right] \\ &= \frac{1}{2\pi}\left[\frac{\pi}{n} + \frac{\pi}{n}\right] \\ &= \frac{1}{2\pi}\left[\frac{2\pi}{n}\right] \\ &= \frac{1}{n} \end{aligned}$$

Substituting in (1),

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{1}{n}\sin nx$$

**Deduction:**

Here $x = \frac{\pi}{2}$ is a point of continuity.

$$\begin{aligned} f\left(\frac{\pi}{2}\right) &= \sum_{n=1}^{\infty} \frac{1}{n} \sin n\frac{\pi}{2} \\ \frac{1}{2}\left(\pi - \frac{\pi}{2}\right) &= \sin\frac{\pi}{2} + \frac{1}{2}\sin\frac{2\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{4}\sin\frac{4\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2}\dots \\ \frac{1}{2}\left(\frac{\pi}{2}\right) &= 1 + 0 + \frac{1}{3}(-1) + 0 + \frac{1}{5}(1) + \dots \\ 1 - \frac{1}{3} + \frac{1}{5} + \dots &= \frac{\pi}{4} \end{aligned}$$

2. Find the Fourier series of $f(x) = (\pi - x)^2$ in $(0, 2\pi)$.Deduce that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \infty = \frac{\pi^2}{6}$

**Solution:**

The Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx ................(1)$$

**To find** $a_0$

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)dx \\ &= \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2 dx \\ &= \frac{1}{\pi}\left[\frac{(\pi - x)^3}{3(-1)}\right]_0^{2\pi} \\ &= \frac{1}{-3\pi}[(-\pi)^3 - \pi^3] \\ &= \frac{1}{-3\pi}[-2\pi^3] \\ &= \frac{2\pi^2}{3} \end{aligned}$$

**To find** $a_n$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx \\ &= \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2 \cos nx dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\pi}\left[(\pi-x)^2\frac{\sin nx}{n} - 2(\pi-x)(-1)\frac{(-\cos nx)}{n^2} + 2(-1)(-1)\frac{(-\sin nx)}{n^3}\right]_0^{2\pi} \\
&= \frac{1}{\pi}\left[(\pi-x)^2\frac{\sin nx}{n} - 2(\pi-x)\frac{\cos nx}{n^2} - 2\frac{\sin nx}{n^3}\right]_0^{2\pi} \\
&= \frac{1}{\pi}\left[0 - 2(-\pi)\frac{1}{n^2} - 0 - \left\{0 - 2\pi\frac{1}{n^2} - 0\right\}\right] \\
&= \frac{1}{\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right] \\
&= \frac{1}{\pi}\left[\frac{4\pi}{n^2}\right] \\
&= \frac{4}{n^2}
\end{aligned}
$$

**To find $b_n$**

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx \\
&= \frac{1}{\pi}\int_0^{2\pi} (\pi-x)^2\sin nx dx \\
&= \frac{1}{\pi}\left[(\pi-x)^2\frac{(-\cos nx)}{n} - 2(\pi-x)(-1)\frac{(-\sin nx)}{n^2} + 2(-1)(-1)\frac{(+\cos nx)}{n^3}\right]_0^{2\pi} \\
&= \frac{1}{\pi}\left[-(\pi-x)^2\frac{\cos nx}{n} - 2(\pi-x)\frac{\sin nx}{n^2} + 2\frac{(\cos nx)}{n^3}\right]_0^{2\pi} \\
&= \frac{1}{\pi}\left[-\pi^2\frac{1}{n} - 0 + \frac{2}{n^3} - \left\{-\pi^2\frac{1}{n} - 0 + \frac{2}{n^3}\right\}\right] \\
&= 0
\end{aligned}
$$

Substituting in (1)

$$\Rightarrow f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx$$

**Deduction:**

Here $x = 0$ is a point of discontinuity at the end point.

$$
\begin{aligned}
\frac{f(0)+f(2\pi)}{2} &= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos 0 \\
\frac{0+0}{2} &= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2} \\
\frac{\pi^2+(-\pi)^2}{2} &= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}
\end{aligned}
$$

$$
\begin{aligned}
4\sum_{n=1}^{\infty}\frac{1}{n^2} &= \pi^2 - \frac{\pi^2}{3} \\
4\sum_{n=1}^{\infty}\frac{1}{n^2} &= \frac{2\pi^2}{3} \\
\sum_{n=1}^{\infty}\frac{1}{n^2} &= \frac{\pi^2}{6} \\
\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots &= \frac{\pi^2}{6}
\end{aligned}
$$

3. Expand in Fourier series of periodic $2\pi$ of $f(x) = x^2$ for $0 < x < 2\pi$. Deduce the sum $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

   Ans: $f(x) = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2} - 4\pi\sum_{n=1}^{\infty}\frac{\sin nx}{n}$

4. Find the Fourier series expansion of $f(x) = \begin{cases} x, & \text{for } 0 < x < \pi; \\ 2\pi - x, & \text{for } \pi < x < 2\pi. \end{cases}$

   Deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\infty = \frac{\pi^2}{8}$.

   **Solution:**

   The Fourier series of $f(x)$ is

$$
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \dots\dots\dots\dots\dots(1)
$$

**To find $a_0$**

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)dx \\
&= \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi}(2\pi - x)dx\right] \\
&= \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi} + \left(2\pi x - \frac{x^2}{2}\right)_{\pi}^{2\pi}\right] \\
&= \frac{1}{\pi}\left[\frac{\pi^2}{2} + 4\pi^2 - 2\pi^2 - \left\{2\pi^2 - \frac{\pi^2}{2}\right\}\right] \\
&= \frac{1}{\pi}\left[\frac{\pi^2}{2} + 2\pi^2 - 2\pi^2 + \frac{\pi^2}{2}\right] \\
&= \frac{1}{\pi}[\pi^2] \\
&= \pi
\end{aligned}
$$

**To find $a_n$**

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx \\
&= \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx dx\right] \\
&= \frac{1}{\pi}\left[\left(x\frac{\sin nx}{n} - (1)\frac{(-\cos nx)}{n^2}\right)_0^{\pi} + \left((2\pi - x)\frac{\sin nx}{n} - (-1)\frac{(-\cos nx)}{n^2}\right)_{\pi}^{2\pi}\right] \\
&= \frac{1}{\pi}\left[\left(x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right)_0^{\pi} + \left((2\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right)_{\pi}^{2\pi}\right] \\
&= \frac{1}{\pi}\left[0 + \frac{(-1)^n}{n^2} - \left\{0 + \frac{1}{n^2}\right\} + 0 - \frac{1}{n^2} - \left\{0 - \frac{(-1)^n}{n^2}\right\}\right] \\
&= \frac{1}{\pi}\left[2\frac{(-1)^n}{n^2} - 2\frac{1}{n^2}\right] \\
&= \frac{2}{n^2\pi}\left[(-1)^n - 1\right] \\
&= \begin{cases} 0, & \text{when 'n' is even;} \\ \frac{-4}{n^2\pi}, & \text{when 'n' is odd.} \end{cases}
\end{aligned}$$

**To find $b_n$**

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx \\
&= \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx dx\right] \\
&= \frac{1}{\pi}\left[\left(x\frac{(-\cos nx)}{n} - (1)\frac{(-\sin nx)}{n^2}\right)_0^{\pi} + \left((2\pi - x)\frac{(-\cos nx)}{n} - (-1)\frac{(-\sin nx)}{n^2}\right)_{\pi}^{2\pi}\right] \\
&= \frac{1}{\pi}\left[\left(-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right)_0^{\pi} + \left(-(2\pi - x)\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right)_{\pi}^{2\pi}\right] \\
&= \frac{1}{\pi}\left[-\pi\frac{(-1)^n}{n} + 0 - \{0 - 0\} - 0 - 0 - \left\{-\pi\frac{(-1)^n}{n} - 0\right\}\right] \\
&= 0
\end{aligned}$$

Substituting in (1)

$$\Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=odd}^{\infty}\frac{-4}{n^2\pi}\cos nx$$

$$= \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\cos nx}{n^2}$$

**Deduction:**

Here $x = 0$ is a point of discontinuity at the end point.

$$\begin{aligned}
\frac{f(0)+f(2\pi)}{2} &= \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\cos 0}{n^2} \\
\frac{0+0}{2} &= \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} \\
0 &= \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right] \\
\frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right] &= \frac{\pi}{2} \\
\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots &= \frac{\pi^2}{8}
\end{aligned}$$

5. Find the Fourier series of $f(x) = x(2\pi - x)$. Deduce $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$.

6. Express $f(x) = x \sin x$ as a Fourier series in $0 \le x \le 2\pi$.

**Solution:**

The Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \ldots\ldots\ldots\ldots\ldots(1)$$

**To find** $a_0$

$$\begin{aligned}
a_0 &= \frac{1}{\pi} \int_0^{2\pi} f(x)dx \\
&= \frac{1}{\pi} \int_0^{2\pi} x \sin x dx \\
&= \frac{1}{\pi} \left[x(-\cos x) - (1)(-\sin x)\right]_0^{2\pi} \\
&= \frac{1}{\pi} \left[-x\cos x + \sin x\right]_0^{2\pi} \\
&= \frac{1}{\pi}[-2\pi - 0] \\
&= -2
\end{aligned}$$

**To find $a_n$**

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\,dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(1+n)x+\sin(1-n)x\right]\,dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(1+n)x-\sin(n-1)x\right]\,dx \\
&= \frac{1}{2\pi}\left[x\left\{\frac{-\cos(n+1)x}{n+1}-\frac{-\cos(n-1)x}{n-1}\right\}-(1)\left\{\frac{-\sin(n+1)x}{(n+1)^2}+\frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi} \\
&= \frac{1}{2\pi}\left[-\frac{x\cos(n+1)x}{n+1}+\frac{x\cos(n-1)x}{n-1}+\frac{\sin(n+1)x}{(n+1)^2}-\frac{\sin(n-1)x}{(n-1)^2}\right]_0^{2\pi} \\
&= \frac{1}{2\pi}\left[-\frac{2\pi}{n+1}+\frac{2\pi}{n-1}\right] \\
&= \frac{1}{2\pi}2\pi\left[-\frac{1}{n+1}+\frac{1}{n-1}\right] \\
&= \frac{-(n-1)+n+1}{(n+1)(n-1)} \\
&= \frac{2}{n^2-1} \qquad \text{Provided } n\neq 1
\end{aligned}$$

**To find $a_1$**

$$\begin{aligned}
a_1 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x\,dx \\
&= \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x\,dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\,dx \\
&= \frac{1}{2\pi}\left[x\left(\frac{-\cos 2x}{2}\right)-(1)\left(\frac{-\sin 2x}{4}\right)\right]_0^{2\pi} \\
&= \frac{1}{2\pi}\left[-\frac{x\cos 2x}{2}+\frac{\sin 2x}{4}\right]_0^{2\pi} \\
&= \frac{1}{2\pi}\left[-\frac{2\pi}{2}\right]
\end{aligned}$$

$$= -\frac{1}{2}$$

**To find $b_n$**

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin nx\,dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} x\left[\cos(n-1)x - \cos(n+1)x\right]\,dx \\
&= \frac{1}{2\pi}\left[x\left\{\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right\} - (1)\left\{\frac{-\cos(n-1)x}{(n-1)^2} - \frac{-\cos(n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi} \\
&= \frac{1}{2\pi}\left[\frac{x\sin(n-1)x}{n-1} - \frac{x\sin(n+1)x}{n+1} + \frac{\cos(n-1)x}{(n-1)^2} - \frac{\cos(n+1)x}{(n+1)^2}\right]_0^{2\pi} \\
&= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \left\{\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2}\right\}\right] \\
&= 0 \qquad\qquad \text{Provided } n \neq 1
\end{aligned}$$

**To find $b_1$**

$$\begin{aligned}
b_1 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin x\,dx \\
&= \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin x\,dx \\
&= \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x\,dx \\
&= \frac{1}{\pi}\int_0^{2\pi} x\frac{1-\cos 2x}{2}dx \\
&= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - (1)\left(\frac{x^2}{2} - \frac{-\cos 2x}{4}\right)\right]_0^{2\pi} \\
&= \frac{1}{2\pi}\left[x^2 - \frac{x\sin 2x}{2} - \frac{x^2}{2} - \frac{\cos 2x}{4}\right]_0^{2\pi} \\
&= \frac{1}{2\pi}\left[4\pi^2 - \frac{4\pi^2}{2} - \frac{1}{4} - \left\{-\frac{1}{4}\right\}\right]
\end{aligned}$$

$$= \frac{1}{2\pi}\left[2\pi^2\right]$$
$$= \pi$$

Substituting in (1)

$$\Rightarrow f(x) = -1 - \frac{1}{2}\cos x + \pi \sin x + \sum_{n=2}^{\infty} \frac{2}{n^2-1}\cos nx$$

7. Find the Fourier series expansion of $f(x) = \begin{cases} \sin x, & \text{for } 0 \le x \le \pi; \\ 0, & \text{for } \pi \le x \le 2\pi. \end{cases}$ Deduce that $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots \infty$.

**Solution:**

The Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

**To find** $a_0$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$
$$= \frac{1}{\pi}\int_0^{\pi} \sin x dx$$
$$= \frac{1}{\pi}\left[-\cos x\right]_0^{\pi}$$
$$= \frac{-1}{\pi}[-1-1]$$
$$= \frac{2}{\pi}$$

**To find** $a_n$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx$$
$$= \frac{1}{\pi}\int_0^{\pi} \sin x \cos nx dx$$

$$= \frac{1}{2\pi}\int_0^{\pi}[\sin(1+n)x+\sin(1-n)x]\,dx$$

$$= \frac{1}{2\pi}\int_0^{\pi}[\sin(1+n)x-\sin(n-1)x]\,dx$$

$$= \frac{1}{2\pi}\left[\frac{-\cos(n+1)x}{n+1}-\frac{-\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{-(-1)^{n+1}}{n+1}+\frac{(-1)^{n-1}}{n-1}-\left\{\frac{-1}{n+1}+\frac{1}{n-1}\right\}\right]$$

$$= \frac{1}{2\pi}\left[\frac{-(-1)^{n+1}}{n+1}+\frac{(-1)^{n+1}}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[(-1)^{n+1}\left\{\frac{-(n-1)+(n+1)}{(n+1)(n-1)}\right\}+\frac{n-1-(n+1)}{(n+1)(n-1)}\right]$$

$$= \frac{1}{2\pi}\left[(-1)^{n+1}\frac{2}{(n^2-1)}+\frac{-2)}{(n^2-1)}\right]$$

$$= \frac{1}{2\pi}\frac{2}{(n^2-1)}[(-1)^{n+1}-1]$$

$$= \frac{1}{(n^2-1)\pi}[(-1)^{n+1}-1]$$

$$= \begin{cases} 0, & \text{when 'n' is odd;} \\ \frac{-2}{(n^2-1)\pi}, & \text{when 'n' is even.} \end{cases} \qquad \text{provided } n \neq 1$$

**To find $a_1$**

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\sin x\cos x dx$$

$$= \frac{1}{2\pi}\int_0^{\pi}\frac{\sin 2x}{2}dx$$

$$= \frac{1}{2\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi}$$

$$= \frac{-1}{4\pi}[1-1]$$

$$= 0$$

**To find $b_n$**

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^\pi f(x)\sin nx dx \\ &= \frac{1}{\pi}\int_0^\pi \sin x \sin nx dx \\ &= \frac{1}{2\pi}\int_0^\pi \left[\cos(n-1)x - \cos(n+1)x\right] dx \\ &= \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^\pi \\ &= 0 \qquad\qquad \text{provided } n \neq 1 \end{aligned}$$

**To find $b_1$**

$$\begin{aligned} b_1 &= \frac{1}{\pi}\int_0^\pi f(x)\sin x dx \\ &= \frac{1}{\pi}\int_0^\pi \sin x \sin x dx \\ &= \frac{1}{\pi}\int_0^\pi \sin^2 x dx \\ &= \frac{1}{\pi}\int_0^\pi \frac{1-\cos 2x}{2} dx \\ &= \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^\pi \\ &= \frac{1}{2\pi}[\pi] \\ &= \frac{1}{2} \end{aligned}$$

Substituting in (1)

$$\begin{aligned} \Rightarrow f(x) &= \frac{\frac{2}{\pi}}{2} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=2,4,\ldots}^{\infty} \frac{1}{n^2-1}\cos nx \\ &= \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n)^2-1}\cos 2nx \\ &= \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \ldots\right] \end{aligned}$$

**Deduction:**

Here $x = 0$ is a point of continuity.

$$\begin{aligned} f(0) &= \frac{1}{\pi} + \frac{1}{2}\sin 0 - \frac{2}{\pi}\left[\frac{1}{2^2-1} + \frac{1}{4^2-1} + \frac{1}{6^2-1} + \dots\right] \\ 0 &= \frac{1}{\pi} - \frac{2}{\pi}\left[\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \dots\right] \\ \frac{2}{\pi}\left[\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \dots\right] &= \frac{1}{\pi} \\ \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \dots\infty &= \frac{1}{2} \end{aligned}$$

**Note:**

1. If $f(-x) = f(x)$ then f(x) is said to be even funtion.

   example: $\cos x, x^2, x^4, |x|, x\sin x, \sin|x|, etc,$

2. If $f(-x) = -f(x)$ then f(x) is said to be odd function.

   example: $\sin x, x, x^3, x\cos x etc,.$

3. If f(x) is an even function then $\int\limits_{-\infty}^{\infty} f(x)dx = 2\int\limits_{0}^{\infty} f(x)dx$

4. If f(x) is an odd function then $\int\limits_{-\infty}^{\infty} f(x)dx = 0$

5. If f(x) is an even function in $(-\pi, \pi)$ then find $a_0, a_n (b_n = 0)$.

6. If f(x) is an odd function in $(-\pi, \pi)$ then find $b_n (a_0 = a_n = 0)$.

8. Find the Fourier series of $f(x) = x^2$ in $(-\pi, \pi)$ of periodicity $2\pi$. Hence deduce that

(i). $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$

(ii). $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}$

(iii). $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{8}$

(iv). $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90}$

**Solution:**

Since $f(x)$ is an even function,

$$b_n = 0.$$

The Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \dots\dots\dots\dots\dots(1)$$

**To find $a_0$**

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= 2\frac{1}{\pi}\int_{0}^{\pi} x^2 dx \\ &= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} \\ &= \frac{2}{3\pi}[\pi^3] \\ &= \frac{2\pi^2}{3} \end{aligned}$$

**To find $a_n$**

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx dx \\ &= \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} - (2x)\frac{(-\cos nx)}{n^2} + 2\frac{(-\sin nx)}{n^3}\right]_0^{\pi} \\ &= \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} + (2x)\frac{\cos nx}{n^2} - 2\frac{\sin nx}{n^3}\right]_0^{\pi} \\ &= \frac{2}{\pi}\left[0 + 2\pi\frac{(-1)^n}{n^2} - 0 - \{0+0-0\}\right] \\ &= \frac{4(-1)^n}{n^2} \end{aligned}$$

Substituting in (1),

$$\begin{aligned} \Rightarrow f(x) &= \frac{2\pi^2}{3\times 2} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx \\ &= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx \end{aligned}$$

**Deduction:**

(i). Here $x = \pi$ is a point of continuity.

$$\begin{aligned} f(\pi) &= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}(-1)^n \\ \pi^2 - \frac{\pi^2}{3} &= 4\sum_{n=1}^{\infty} \frac{1}{n^2} \\ \frac{2\pi^2}{3} &= 4\sum_{n=1}^{\infty} \frac{1}{n^2} \\ \frac{\pi^2}{6} &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots(3) \end{aligned}$$

(ii). Here $x = 0$ is a point of continuity.

$$\begin{aligned} f(0) &= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \\ 0 - \frac{\pi^2}{3} &= 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \\ -\frac{\pi^2}{12} &= \frac{(-1)}{1^2} + \frac{1}{2^2} + \frac{(-1)}{3^2} + \ldots \\ -\frac{\pi^2}{12} &= -\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots\right] \\ \frac{\pi^2}{12} &= \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots\ldots(4) \end{aligned}$$

(iii). (3) + (4)

$$\begin{aligned} 2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right] &= \frac{\pi^2}{6} + \frac{\pi^2}{12} \\ &= \frac{3\pi^2}{12} \\ &= \frac{\pi^2}{4} \\ \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots &= \frac{\pi^2}{8} \end{aligned}$$

(iv). By parseval's identity,

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx &= \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \\ \frac{1}{2\pi}\int_{-\pi}^{\pi} x^4 dx &= \frac{\frac{4\pi^4}{9}}{4} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{16}{n^4} \end{aligned}$$

$$\begin{aligned}
\frac{2}{2\pi}\int_0^\pi x^4 dx &= \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4} \\
\frac{1}{\pi}\left[\frac{x^5}{5}\right]_0^\pi &= \frac{4\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4} \\
\frac{1}{\pi}\left[\frac{\pi^5}{5}\right] &= \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4} \\
\frac{\pi^4}{5} &= \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4} \\
8\sum_{n=1}^{\infty}\frac{1}{n^4} &= \frac{\pi^4}{5} - \frac{\pi^4}{9} \\
8\sum_{n=1}^{\infty}\frac{1}{n^4} &= \frac{4\pi^4}{45} \\
\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots &= \frac{\pi^4}{90}
\end{aligned}$$

9. Obtain the Fourier series to represent the function $f(x) = |x|, , -\pi < x < \pi$ and deduce $\sum_0^\infty \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

**Solution:**

Since $f(x)$ is an even function$\Rightarrow b_n = 0$.

The Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

**To find** $a_0$

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\
&= \frac{2}{\pi}\int_0^\pi x dx \\
&= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^\pi \\
&= \frac{1}{\pi}[\pi^2] \\
&= \pi
\end{aligned}$$

**To find $a_n$**

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} x\cos nx dx \\
&= \frac{2}{\pi}\left[x\frac{\sin nx}{n} - (1)\frac{(-\cos nx)}{n^2}\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[0 + \frac{(-1)^n}{n^2} - \left\{0 + \frac{1}{n^2}\right\}\right] \\
&= \frac{2}{\pi n^2}[(-1)^n - 1] \\
&= \begin{cases} \dfrac{-4}{n^2\pi}, & \text{when 'n' is odd;} \\ 0, & \text{when 'n' is even.} \end{cases}
\end{aligned}$$

Substituting in (1),

$$\Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1,3,5,\ldots}^{\infty} \frac{-4}{n^2\pi}\cos nx$$

$$\Rightarrow f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2}\cos nx$$

**Deduction:**

Here $x = 0$ is a point of continuity.

$$\begin{aligned}
f(0) &= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2}\cos 0 \\
0 &= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} \\
\frac{4}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} &= \frac{\pi}{2} \\
\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} &= \frac{\pi^2}{8} \\
\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots &= \frac{\pi^2}{8}
\end{aligned}$$

10. Find the Fourier series of $f(x) = x$ in $(-\pi, \pi)$.

11. Find the Fourier series of $f(x) = x + x^2$ in $(-\pi, \pi)$ of periodicity $2\pi$

**Solution:**

Given: f(x) is neither odd nor even.

The Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx................(1)$$

**To find** $a_0$

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)dx \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} xdx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2dx\right] \\ &= \frac{1}{\pi}\left[0 + 2\int_{0}^{\pi} x^2dx\right] \\ &= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} \\ &= \frac{2}{3\pi}[\pi^3] \\ &= \frac{2\pi^2}{3} \end{aligned}$$

**To find** $a_n$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)\cos nxdx \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\cos nxdx + \int_{-\pi}^{\pi} x^2\cos nxdx\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\pi}\left[0 + 2\int_0^{\pi} x^2 \cos nx dx\right] \\
&= \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} - (2x)\frac{(-\cos nx)}{n^2} + 2\frac{(-\sin nx)}{n^3}\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} + (2x)\frac{\cos nx}{n^2} - 2\frac{\sin nx}{n^3}\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[0 + 2\pi\frac{(-1)^n}{n^2} - 0 - \{0+0-0\}\right] \\
&= \frac{4(-1)^n}{n^2}
\end{aligned}
$$

**To find $b_n$**

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)\sin nx dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\sin nx dx + \int_{-\pi}^{\pi} x^2 \sin nx dx\right] \\
&= \frac{1}{\pi}\left[2\int_0^{\pi} x\sin nx dx + 0\right] \\
&= \frac{2}{\pi}\left[x\frac{(-\cos nx)}{n} - (1)\frac{(-\sin nx)}{n^2}\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[-\pi\frac{(-1)^n}{n} - 0 - \{0-0\}\right] \\
&= -\frac{2(-1)^n}{n}
\end{aligned}
$$

Substituting in (1),

$$
\begin{aligned}
\Rightarrow f(x) &= \frac{2\pi^2}{3\times 2} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx - \sum_{n=1}^{\infty}\frac{2(-1)^n}{n}\sin nx \\
&= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx
\end{aligned}
$$

**Note:**

Let $f(x) = \begin{cases} \phi_1(x), & -\pi \le x \le 0; \\ \phi_2(x), & \text{for } 0 \le x \le \pi. \end{cases}$

If $\phi_1(-x) = \phi_2(x)$ then $f(x)$ is even.

If $\phi_1(-x) = -\phi_2(x)$ then $f(x)$ is odd.

12. If $f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi \le x \le 0; \\ 1 - \frac{2x}{\pi}, & \text{for } 0 \le x \le \pi. \end{cases}$ show that $f(x) = \frac{8}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right]$

Deduce that $\sum_1^\infty \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

**Solution:**

Given:

$$\begin{aligned} \phi_1(x) &= 1 + \frac{2x}{\pi} \\ \phi_1(-x) &= 1 - \frac{2x}{\pi} \\ &= \phi_2(x) \end{aligned}$$

$$\Rightarrow \text{ f(x) is an even function } \Rightarrow b_n = 0.$$

The Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \ldots\ldots\ldots\ldots\ldots(1)$$

**To find** $a_0$

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)dx \\ &= \frac{2}{\pi}\left[x - \frac{2x^2}{2\pi}\right]_0^{\pi} \\ &= \frac{2}{\pi}[\pi - \pi] \\ &= 0 \end{aligned}$$

**To find $a_n$**

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \\
&= \frac{2}{\pi}\int_{0}^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx dx \\
&= \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\frac{\sin nx}{n} - \left(-\frac{2}{\pi}\right)\frac{(-\cos nx)}{n^2}\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\frac{\sin nx}{n} - \frac{2}{\pi}\frac{\cos nx}{n^2}\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[-\frac{2(-1)^n}{\pi n^2} - \left\{-\frac{2}{\pi n^2}\right\}\right] \\
&= \frac{2}{\pi}\frac{2}{\pi n^2}[1-(-1)^n] \\
&= \frac{4}{\pi^2 n^2}[1-(-1)^n] \\
&= \begin{cases} \frac{8}{\pi^2 n^2}, & \text{when 'n' is odd;} \\ 0, & \text{when 'n' is even.} \end{cases}
\end{aligned}$$

Substituting in (1),

$$\Rightarrow f(x) = \sum_{n=1,3,5,\dots}^{\infty} \frac{8}{\pi^2 n^2}\cos nx$$

$$\Rightarrow f(x) = \frac{8}{\pi^2}\sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2}\cos nx$$

**Deduction:**

Here $x = 0$ is a point of continuity.

$$\begin{aligned}
1 &= \frac{8}{\pi^2}\sum_{n=1,3,5,\dots}^{\infty}\frac{1}{n^2} \\
\sum_{n=1,3,5,\dots}^{\infty}\frac{1}{n^2} &= \frac{\pi^2}{8} \\
\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots &= \frac{\pi^2}{8}
\end{aligned}$$

13. Find the Fourier series expansion of the periodic function $f(x)$ of the period $2\pi$ defined by

$$f(x) = \begin{cases} \pi + x, & \text{for } -\pi \le x < 0; \\ \pi - x, & \text{for } 0 < x \le \pi. \end{cases}$$ Deduce that $\sum_{1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

**Solution:**

Given:

$$\begin{aligned} \phi_1(x) &= \pi + x \\ \phi_1(-x) &= \pi - x \\ &= \phi_2(x) \end{aligned}$$

$\Rightarrow f(x)$ is an even function$\Rightarrow b_n = 0$.

The Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx................(1)$$

**To find** $a_0$

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} (\pi - x)dx \\ &= \frac{2}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi} \\ &= \frac{2}{\pi}\left[\pi^2 - \frac{\pi^2}{2} - 0 - 0\right] \\ &= \frac{1}{\pi}\left[\frac{\pi^2}{2}\right] \\ &= \pi \end{aligned}$$

**To find** $a_n$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} (\pi - x)\cos nx dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{\pi}\left[(\pi - x)\frac{\sin nx}{n} - (-1)\frac{(-\cos nx)}{n^2}\right]_0^\pi \\
&= \frac{2}{\pi}\left[(\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]_0^\pi \\
&= \frac{2}{\pi}\left[0 - \frac{(-1)^n}{n^2} - \left\{0 - \frac{1}{n^2}\right\}\right] \\
&= \frac{2}{\pi n^2}[1 - (-1)^n] \\
&= \begin{cases} \frac{4}{n^2\pi}, & \text{when 'n' is odd;} \\ 0, & \text{when 'n' is even.} \end{cases}
\end{aligned}
$$

Substituting in (1),

$$\Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1,3,5,\ldots}^{\infty} \frac{4}{n^2\pi}\cos nx$$

$$\Rightarrow f(x) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2\pi}\cos nx$$

**Deduction:**

Here $x = 0$ is a point of discontinuity at the mid point.

$$
\begin{aligned}
\frac{f(0-) + f(0+)}{2} &= f(x) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2}\cos 0 \\
\frac{\pi + \pi}{2} &= \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} \\
\pi - \frac{\pi}{2} &= \frac{4}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} \\
\frac{4}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} &= \frac{\pi}{2} \\
\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots &= \frac{\pi^2}{8}
\end{aligned}
$$

14. If $f(x) = \begin{cases} 1 - x, & \text{for } -\pi \le x \le 0; \\ 1 + x, & \text{for } 0 \le x \le \pi. \end{cases}$ find the Fourier series of $f(x)$ and hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$.

15. Obtain the Fourier series expansion of $f(x)$ given by $f(x) = \begin{cases} -1 + x, & \text{for } -\pi \le x \le 0; \\ 1 + x, & \text{for } 0 \le x \le \pi. \end{cases}$

and hence deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8}$.

**Solution:**

Given:

$$\begin{aligned}
\phi_1(x) &= -1+x \\
\phi_1(-x) &= -1-x \\
&= -(1+x) \\
&= -\phi_2(x) \\
\Rightarrow \text{ f(x) is an odd function } &\Rightarrow a_0=a_n=0.
\end{aligned}$$

The Fourier series of $f(x)$ is

$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx \ldots\ldots\ldots\ldots\ldots..(1)$$

**To find** $b_n$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \\
&= \frac{2}{\pi}\int_{0}^{\pi}(1+x)\sin nx dx \\
&= \frac{2}{\pi}\left[(1+x)\frac{(-\cos nx)}{n}-(1)\frac{(-\sin nx)}{n^2}\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[-(1+x)\frac{\cos nx}{n}+\frac{\sin nx}{n^2}\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[-(1+\pi)\frac{(-1)^n}{n}+0-\left\{-\frac{1}{n}+0\right\}\right] \\
&= \frac{2}{n\pi}\left[-(1+\pi)(-1)^n+1\right]
\end{aligned}$$

Sub in (1)

$$\Rightarrow f(x) = \sum_{n=1}^{\infty}\frac{2}{n\pi}\left[-(1+\pi)(-1)^n+1\right]\sin nx$$

16. Express $f(x)=x\sin x$ as a Fourier series in $(-\pi,\pi)$.

**Solution:**

Given:

$$\begin{aligned} f(x) &= x\sin x \\ f(-x) &= -x\sin(-x) \\ &= x\sin x \\ &= f(x) \\ \Rightarrow \text{f(x) is an even function} &\Rightarrow b_n = 0. \end{aligned}$$

The Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx .................(1)$$

**To find $a_0$**

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} x\sin x dx \\ &= \frac{2}{\pi}\left[x(-\cos x) - (1)(-\sin x)\right]_0^{\pi} \\ &= \frac{2}{\pi}\left[-x\cos x + \sin x\right]_0^{\pi} \\ &= \frac{2}{\pi}[-\pi(-1) - 0 - \{-0 - 0\}] \\ &= 2 \end{aligned}$$

**To find $a_n$**

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} x\sin x\cos nx dx \\ &= \frac{2}{2\pi}\int_{0}^{\pi} x\left[\sin(1+n)x + \sin(1-n)x\right] dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\pi}\int_{0}^{\pi} x\left[\sin(1+n)x - \sin(n-1)x\right]dx \\
&= \frac{1}{\pi}\left[x\left\{\frac{-\cos(n+1)x}{n+1} - \frac{-\cos(n-1)x}{n-1}\right\} - (1)\left\{\frac{-\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{\pi} \\
&= \frac{1}{\pi}\left[-\frac{x\cos(n+1)x}{n+1} + \frac{x\cos(n-1)x}{n-1} + \frac{\sin(n+1)x}{(n+1)^2} - \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{\pi} \\
&= \frac{1}{\pi}\left[-\frac{\pi(-1)^{n+1}}{n+1} + \frac{\pi(-1)^{n-1}}{n-1}\right] \\
&= \frac{1}{\pi}\pi(-1)^{n+1}\left[-\frac{1}{n+1} + \frac{1}{n-1}\right] \\
&= (-1)^{n+1}\left[\frac{-(n-1)+n+1}{(n+1)(n-1)}\right] \\
&= \frac{2(-1)^{n+1}}{(n^2-1)} \qquad \text{provided } n \neq 1
\end{aligned}
$$

**To find $a_1$**

$$
\begin{aligned}
a_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} x\sin x\cos x dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} x\frac{\sin 2x}{2} dx \\
&= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - (1)\left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi} \\
&= \frac{1}{\pi}\left[-\frac{x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{\pi} \\
&= \frac{1}{\pi}\left[-\frac{\pi}{2}\right] \\
&= -\frac{1}{2}
\end{aligned}
$$

Substituting in (1)

$$\Rightarrow f(x) = 1 - \frac{1}{2}\cos x + \pi\sin x + \sum_{n=2}^{\infty}\frac{2(-1)^{n+1}}{n^2-1}\cos nx.$$

17. Find the Fourier series expansion of the periodic function $f(x)$ of the period $2\pi$ defined by $f(x) = \begin{cases} -k, & \text{for } -\pi < x < 0; \\ k, & \text{for } 0 < x < \pi. \end{cases}$ Deduce that $1 - \frac{1}{3} + \frac{1}{5} + \ldots = \frac{\pi}{4}$.

**Solution:**

Given:

$$\begin{aligned}\phi_1(x) &= -k\\ \phi_1(-x) &= -k\\ &= -\phi_2(x)\end{aligned}$$

$\Rightarrow f(x)$ is an odd function, $a_0 = a_n = 0$.

The Fourier series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \ldots\ldots\ldots\ldots\ldots.(1)$$

**To find $b_n$**

$$\begin{aligned}b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx\\ &= \frac{2}{\pi}\int_{0}^{\pi} k\sin nx dx\\ &= \frac{2k}{\pi}\int_{0}^{\pi} \sin nx dx\\ &= \frac{2k}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi}\\ &= \frac{-2k}{n\pi}[(-1)^n - 1]\\ &= \begin{cases}\frac{4k}{n\pi}, & \text{when 'n' is odd;}\\ 0, & \text{when 'n' is even.}\end{cases}\end{aligned}$$

Substituting in (1),

$$\Rightarrow f(x) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{4k}{n\pi}\sin nx$$

**Deduction:**

Here $x = \frac{\pi}{2}$ is a point of continuity.

$$\begin{aligned} f\left(\frac{\pi}{2}\right) &= \frac{4k}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \sin nx \\ k &= \frac{4k}{\pi}\left[\sin\frac{\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2}\ldots\right] \\ \frac{\pi}{4} &= 1 + \frac{1}{3}(-1) + \frac{1}{5}(1) + \ldots \\ 1 - \frac{1}{3} + \frac{1}{5} - \ldots &= \frac{\pi}{4} \end{aligned}$$

18. If $f(x) = \begin{cases} -1, & \text{for } -\pi < x < 0; \\ 1, & \text{for } 0 < x < \pi. \end{cases}$ and $f(x+2\pi) = f(x)$ for all x, find the Fourier series for f(x).

19. Obtain the Fourier series of the periodic function defined by $f(x) = \begin{cases} -\pi, & \text{for } -\pi < x < 0; \\ x, & \text{for } 0 < x < \pi. \end{cases}$

    Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$

20. Find the Fourier expansion of $f(x) = x$ in the interval $(-\pi, \pi)$.

21. Find the Fourier series expansion of $f(x) = \begin{cases} 0, & \text{for } -\pi \le x \le 0; \\ \sin x, & \text{for } 0 \le x \le \pi. \end{cases}$ Deduce that

    $(i) \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots\infty = \frac{1}{2}.$

    $(ii) \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \ldots\infty \frac{\pi - 2}{4}.$

**Solution:**

The Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

**To find** $a_0$

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= \frac{1}{\pi}\int_{0}^{\pi} \sin x dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\pi}\left[-\cos x\right]_0^{\pi} \\
&= \frac{-1}{\pi}[-1-1] \\
&= \frac{2}{\pi}
\end{aligned}
$$

**To find $a_n$**

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} \sin x \cos nx dx \\
&= \frac{1}{2\pi}\int_{0}^{\pi} \left[\sin(1+n)x + \sin(1-n)x\right] dx \\
&= \frac{1}{2\pi}\int_{0}^{\pi} \left[\sin(1+n)x - \sin(n-1)x\right] dx \\
&= \frac{1}{2\pi}\left[\frac{-\cos(n+1)x}{n+1} - \frac{-\cos(n-1)x}{n-1}\right]_0^{\pi} \\
&= \frac{1}{2\pi}\left[\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi} \\
&= \frac{1}{2\pi}\left[\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} - \left\{\frac{-1}{n+1} + \frac{1}{n-1}\right\}\right] \\
&= \frac{1}{2\pi}\left[\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n+1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] \\
&= \frac{1}{2\pi}\left[(-1)^{n+1}\left\{\frac{-(n-1)+(n+1)}{(n+1)(n-1)}\right\} + \frac{n-1-(n+1)}{(n+1)(n-1)}\right] \\
&= \frac{1}{2\pi}\left[(-1)^{n+1}\frac{2}{(n^2-1)} + \frac{-2}{(n^2-1)}\right] \\
&= \frac{1}{2\pi}\frac{2}{(n^2-1)}\left[(-1)^{n+1} - 1\right] \\
&= \frac{1}{(n^2-1)\pi}\left[(-1)^{n+1} - 1\right] \\
&= \begin{cases} 0, & \text{when 'n' is odd;} \\ \dfrac{-2}{(n^2-1)\pi}, & \text{when 'n' is even.} \end{cases} \qquad \text{provided } n \neq 1
\end{aligned}
$$

**To find $a_1$**

$$
a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x dx
$$

$$
\begin{aligned}
&= \frac{1}{\pi}\int_{0}^{\pi} \sin x \cos x dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} \frac{\sin 2x}{2} dx \\
&= \frac{1}{2\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi} \\
&= \frac{-1}{4\pi}[1-1] \\
&= 0
\end{aligned}
$$

**To find** $b_n$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} \sin x \sin nx dx \\
&= \frac{1}{2\pi}\int_{0}^{\pi} [\cos(n-1)x - \cos(n+1)x] \\
&= \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^{\pi} \\
&= 0 \qquad \text{provided } n \neq 1
\end{aligned}
$$

**To find** $b_1$

$$
\begin{aligned}
b_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin x dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} \sin x \sin x dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} \sin^2 x dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} \frac{1-\cos 2x}{2} dx \\
&= \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} \\
&= \frac{1}{2\pi}[\pi]
\end{aligned}
$$

$$= \frac{1}{2}$$

Substituting in (1)

$$\Rightarrow f(x) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=2,4,\ldots}^{\infty} \frac{1}{n^2-1}\cos nx + \frac{1}{2}\sin x$$

$$f(x) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=2,4,\ldots}^{\infty} \frac{1}{(n-1)(n+1)}\cos nx + \frac{1}{2}\sin x$$

**Deduction:**

Here $x = 0$ is a point of continuity.

$$f(0) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=2,4,\ldots}^{\infty} \frac{1}{(n-1)(n+1)}(1) + \frac{1}{2}(0)$$

$$0 = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=2,4,\ldots}^{\infty} \frac{1}{(n-1)(n+1)}$$

$$\frac{2}{\pi}\sum_{n=2,4,\ldots}^{\infty} \frac{1}{(n-1)(n+1)} = \frac{1}{\pi}$$

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots\infty = \frac{1}{2}$$

Here $x = \frac{\pi}{2}$ is a point of continuity.

$$f\left(\frac{\pi}{2}\right) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=2,4,\ldots}^{\infty} \frac{1}{(n-1)(n+1)}\cos\frac{n\pi}{2} + \frac{1}{2}(1)$$

$$1 - \frac{1}{2} - \frac{1}{\pi} = -\frac{2}{\pi}\left[\frac{1}{1\cdot 3}(-1) + \frac{1}{3\cdot 5}(1) + \frac{1}{5\cdot 7}(-1) + \ldots\right]$$

$$\frac{1}{2} - \frac{1}{\pi} = \frac{2}{\pi}\left[\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \ldots\right]$$

$$\frac{\pi-2}{2\pi}\frac{\pi}{2} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \ldots$$

$$\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \ldots\infty = \frac{\pi-2}{4}$$

## $l$ Type

22. Find the Fourier series expansion of $f(x) = \begin{cases} x, & \text{for } 0 < x \leq l; \\ 2l - x, & \text{for } l \leq x < 2l. \end{cases}$ Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \infty = \frac{\pi^2}{8}$.

**Solution:**

The Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{................(1)}$$

**To find** $a_0$

$$\begin{aligned}
a_0 &= \frac{1}{l}\int_0^{2l} f(x)dx \\
&= \frac{1}{l}\left[\int_0^{l} xdx + \int_l^{2l} (2l-x)dx\right] \\
&= \frac{1}{l}\left[\left(\frac{x^2}{2}\right)_0^l + \left(2lx - \frac{x^2}{2}\right)_l^{2l}\right] \\
&= \frac{1}{l}\left[\left(\frac{l^2}{2}\right)_0^l + \left(4l^2 - 2l^2 - \left\{2l^2 - \frac{1^2}{2}\right\}\right)\right] \\
&= \frac{1}{l}\left[\left(\frac{l^2}{2}\right) + 2l^2 - 2l^2 + \frac{l^2}{2}\right] \\
&= \frac{1}{l}[l^2] \\
&= l
\end{aligned}$$

**To find** $a_n$

$$\begin{aligned}
a_n &= \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}dx \\
&= \frac{1}{l}\left[\int_0^{l} x\cos\frac{n\pi x}{l}dx + \int_l^{2l} (2l-x)\cos\frac{n\pi x}{l}dx\right] \\
&= \frac{1}{l}\left[\left(x\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}} - (1)\frac{(-\cos\frac{n\pi x}{l})}{\left(\frac{n\pi}{l}\right)^2}\right)_0^l + \left((2l-x)\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}} - (-1)\frac{(-\cos\frac{n\pi x}{l})}{\left(\frac{n\pi}{l}\right)^2}\right)_l^{2l}\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{l}\left[\left(x\frac{l}{n\pi}\sin\frac{n\pi x}{l}+\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right)_0^l+\left((2l-x)\frac{l}{n\pi}\sin\frac{n\pi x}{l}-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right)_l^{2l}\right]\\
&= \frac{1}{l}\left[0+\frac{l^2}{n^2\pi^2}(-1)^n-\left\{0+\frac{l^2}{n^2\pi^2}\right\}+0-\frac{l^2}{n^2\pi^2}-\left\{0-\frac{l^2}{n^2\pi^2}(-1)^n\right\}\right]\\
&= \frac{1}{l}\frac{l^2}{n^2\pi^2}\left[2(-1)^n-2\right]\\
&= \frac{2l}{n^2\pi^2}[(-1)^n-1]\\
&= \begin{cases} 0, & \text{when 'n' is even;}\\ \frac{-4l}{n^2\pi^2}, & \text{when 'n' is odd.}\end{cases}
\end{aligned}$$

**To find $b_n$**

$$\begin{aligned}
b_n &= \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}dx\\
&= \frac{1}{l}\left[\int_0^l x\sin\frac{n\pi x}{l}dx+\int_l^{2l}(2l-x)\sin\frac{n\pi x}{l}dx\right]\\
&= \frac{1}{l}\left[\left(x\frac{(-\cos\frac{n\pi x}{l})}{\frac{n\pi}{l}}-(1)\frac{(-\sin\frac{n\pi x}{l})}{\left(\frac{n\pi}{l}\right)^2}\right)_0^l+\left((2l-x)\frac{(-\cos\frac{n\pi x}{l})}{\frac{n\pi}{l}}-(-1)\frac{(-\sin\frac{n\pi x}{l})}{\left(\frac{n\pi}{l}\right)^2}\right)_l^{2l}\right]\\
&= \frac{1}{l}\left[\left(-x\frac{l}{n\pi}\cos\frac{n\pi x}{l}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)_0^l+\left(-(2l-x)\frac{l}{n\pi}\cos\frac{n\pi x}{l}-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)_l^{2l}\right]\\
&= \frac{1}{l}\left[-l\frac{l}{n\pi}(-1)^n+0-\{0+0\}-0-0-\left\{-l\frac{l}{n\pi}(-1)^n-0\right\}\right]\\
&= 0
\end{aligned}$$

Substituting in (1)

$$\begin{aligned}
\Rightarrow f(x) &= \frac{l}{2}+\sum_{n=odd}^{\infty}\frac{-4l}{n^2\pi^2}\cos\frac{n\pi x}{l}\\
&= \frac{l}{2}-\frac{4l}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{\cos\frac{n\pi x}{l}}{n^2}
\end{aligned}$$

**Deduction:**

Here $x=0$ is a point of discontinuity at the end point.

$$\frac{f(0)+f(2\pi)}{2} = \frac{l}{2}-\frac{4l}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}(1)$$

$$\begin{aligned}
\frac{0+0}{2} &= \frac{l}{2} - \frac{4l}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2} \\
0 &= \frac{l}{2} - \frac{4l}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right] \\
\frac{4l}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right] &= \frac{l}{2} \\
\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots &= \frac{\pi^2}{8}
\end{aligned}$$

23. Find the Fourier series expansion of $f(x) = \begin{cases} l-x, & \text{for } 0<x<l; \\ 0, & \text{for } l<x<2l. \end{cases}$ Deduce that $\sum_0^{\infty}\frac{1}{(2n+1)^2}$.

**Solution:**

The Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \ldots\ldots\ldots\ldots\ldots(1)$$

**To find $a_0$**

$$\begin{aligned}
a_0 &= \frac{1}{l}\int_0^{2l} f(x)dx \\
&= \frac{1}{l}\int_0^{l}(l-x)dx \\
&= \frac{1}{l}\left[lx - \frac{x^2}{2}\right]_0^l \\
&= \frac{1}{l}\left[l^2 - \frac{l^2}{2} - \{0-0\}\right] \\
&= \frac{1}{l}\left[\frac{l^2}{2}\right] \\
&= \frac{l}{2}
\end{aligned}$$

**To find $a_n$**

$$\begin{aligned}
a_n &= \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}dx \\
&= \frac{1}{l}\int_0^{l}(l-x)\cos\frac{n\pi x}{l}dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{l}\left[(l-x)\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}} - (-1)\frac{(-\cos\frac{n\pi x}{l})}{\left(\frac{n\pi}{l}\right)^2}\right]_0^l \\
&= \frac{1}{l}\left[(l-x)\frac{l}{n\pi}\sin\frac{n\pi x}{l} - \frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right]_0^l \\
&= \frac{1}{l}\left[0 - \frac{l^2}{n^2\pi^2}(-1)^n - \left\{0 - \frac{l^2}{n^2\pi^2}\right\}\right] \\
&= \frac{1}{l}\left[-\frac{l^2}{n^2\pi^2}(-1)^n + \frac{l^2}{n^2\pi^2}\right] \\
&= \frac{1}{l}\frac{l^2}{n^2\pi^2}\left[1-(-1)^n\right] \\
&= \frac{l}{n^2\pi^2}\left[1-(-1)^n\right] \\
&= \begin{cases} 0, & \text{when 'n' is even;} \\ \frac{2l}{n^2\pi^2}, & \text{when 'n' is odd.} \end{cases}
\end{aligned}
$$

**To find $b_n$**

$$
\begin{aligned}
b_n &= \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}dx \\
&= \frac{1}{l}\int_0^{l}(l-x)\sin\frac{n\pi x}{l}dx \\
&= \frac{1}{l}\left[(l-x)\frac{(-\cos\frac{n\pi x}{l})}{\frac{n\pi}{l}} - (-1)\frac{(-\sin\frac{n\pi x}{l})}{\left(\frac{n\pi}{l}\right)^2}\right]_0^l \\
&= \frac{1}{l}\left[-(l-x)\frac{l}{n\pi}\cos\frac{n\pi x}{l} - \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_0^l \\
&= \frac{1}{l}\left[0 - 0 - \left\{-l\frac{l}{n\pi}(1) - 0\right\}\right] \\
&= \frac{l}{n\pi}
\end{aligned}
$$

Substituting in (1)

$$
\begin{aligned}
\Rightarrow f(x) &= \frac{\frac{l}{2}}{2} + \sum_{n=odd}^{\infty}\frac{2l}{n^2\pi^2}\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty}\frac{l}{n\pi}\sin\frac{n\pi x}{l} \\
&= \frac{l}{4} + \frac{2l}{\pi^2}\sum_{n=n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l} + \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l}
\end{aligned}
$$

**Deduction:**

Here $x = 0$ is a point of discontinuity at the end point.

$$\begin{aligned}
\frac{f(0)+f(2\pi)}{2} &= \frac{l}{4}+\frac{2l}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}(1) \\
\frac{l+0}{2} &= \frac{l}{4}+\frac{2l}{\pi^2}\sum_{n=n=1,3,5,\ldots}^{\infty}\frac{1}{n^2} \\
\frac{l}{2}-\frac{l}{4} &= \frac{2l}{\pi^2}\sum_{n=n=1,3,5,\ldots}^{\infty}\frac{1}{n^2} \\
\frac{l}{4} &= \frac{2l}{\pi^2}\sum_{n=n=1,3,5,\ldots}^{\infty}\frac{1}{n^2} \\
\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots &= \frac{\pi^2}{8}
\end{aligned}$$

24. Find the Fourier series of periodicity 3 for $f(x) = 2x - x^2$ in $0 < x < 3$.

**Solution:**

Let $2l = 3 \Rightarrow l = \frac{3}{2}$

The Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}\ldots\ldots\ldots\ldots\ldots(1)$$

**To find $a_0$**

$$\begin{aligned}
a_0 &= \frac{1}{l}\int_0^{2l} f(x)dx \\
&= \frac{1}{\frac{3}{2}}\int_0^{3}(2x-x^2)dx \\
&= \frac{2}{3}\left[2\frac{x^2}{2}-\frac{x^3}{3}\right]_0^3 \\
&= \frac{2}{3}[9-9] \\
&= 0
\end{aligned}$$

**To find $a_n$**

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}dx$$

$$
\begin{aligned}
&= \frac{1}{\frac{3}{2}}\int_0^3 (2x-x^2)\cos\frac{n\pi x}{\frac{3}{2}}dx \\
&= \frac{2}{3}\int_0^3 (2x-x^2)\cos\frac{2n\pi x}{3}dx \\
&= \frac{2}{3}\left[(2x-x^2)\frac{\sin\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}-(2-2x)\frac{(-\cos\frac{2n\pi x}{3})}{\left(\frac{2n\pi}{3}\right)^2}+(-2)\frac{(-\sin\frac{2n\pi x}{3})}{\left(\frac{2n\pi}{3}\right)^3}\right]_0^3 \\
&= \frac{2}{3}\left[(2x-x^2)\frac{3}{2n\pi}\sin\frac{2n\pi x}{3}+(2-2x)\left(\frac{3}{2n\pi}\right)^2\cos\frac{2n\pi x}{3}+2\left(\frac{3}{2n\pi}\right)^3\sin\frac{2n\pi x}{3}\right]_0^3 \\
&= \frac{2}{3}\left[0-4\left(\frac{3}{2n\pi}\right)^2+0-\left\{0+2\left(\frac{3}{2n\pi}\right)^2+0\right\}\right] \\
&= \frac{2}{3}\left[-\frac{9}{n^2\pi^2}-\frac{9}{2n^2\pi^2}\right] \\
&= \frac{2}{3}\left[-\frac{36}{4n^2\pi^2}-\frac{18}{4n^2\pi^2}\right] \\
&= \frac{2}{3}\left[-\frac{54}{4n^2\pi^2}\right] \\
&= -\frac{9}{n^2\pi^2}
\end{aligned}
$$

**To find $b_n$**

$$
\begin{aligned}
b_n &= \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}dx \\
&= \frac{1}{\frac{3}{2}}\int_0^3 (2x-x^2)\sin\frac{n\pi x}{\frac{3}{2}}dx \\
&= \frac{1}{\frac{3}{2}}\int_0^3 (2x-x^2)\sin\frac{2n\pi x}{3}dx \\
&= \frac{2}{3}\left[(2x-x^2)\frac{(-\cos\frac{2n\pi x}{3})}{\frac{2n\pi}{3}}-(2-2x)\frac{(-\sin\frac{2n\pi x}{3})}{\left(\frac{2n\pi}{3}\right)^2}+(-2)\frac{\cos\frac{2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^3}\right]_0^3 \\
&= \frac{2}{3}\left[-(2x-x^2)\frac{3}{2n\pi}\cos\frac{2n\pi x}{3}+(2-2x)\left(\frac{3}{2n\pi}\right)^2\sin\frac{2n\pi x}{3}-2\left(\frac{3}{2n\pi}\right)^3\cos\frac{2n\pi x}{3}\right]_0^3 \\
&= \frac{2}{3}\left[-(-3)\frac{3}{2n\pi}+0-2\frac{27}{8n^3\pi^3}-\left\{0+0-2\frac{27}{8n^3\pi^3}\right\}\right] \\
&= \frac{2}{3}\left[\frac{9}{2n\pi}\right] \\
&= \frac{3}{n\pi}
\end{aligned}
$$

Sub in (1)

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{-9}{n^2\pi^2} \cos \frac{n\pi x}{\frac{3}{2}} + \sum_{n=1}^{\infty} \frac{3}{n\pi} \sin \frac{n\pi x}{\frac{3}{2}}$$

$$= -\frac{9}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \frac{2n\pi x}{3} + \frac{3}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{2n\pi x}{3}$$

25. Find the Fourier series of periodicity 3 for $f(x) = 2x - x^2$ in $0 < x < 2$.

26. Find the Fourier Series for $f(x) = \begin{cases} x, & \text{for } 0 \leq x \leq 3; \\ 6 - x, & \text{for } 3 \leq x \leq 6. \end{cases}$

27. Expand $f(x) = x - x^2$ in $-l < x < l$ using Fouries Series.

**Solution:**

Given: f(x) is neither odd nor even.

The Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \ldots\ldots\ldots\ldots\ldots..(1)$$

**To find** $a_0$

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x)dx$$

$$= \frac{1}{l} \int_{-l}^{l} (x - x^2)dx$$

$$= \frac{1}{l} \left[ \int_{-l}^{l} xdx - \int_{-l}^{l} x^2dx \right]$$

$$= \frac{1}{l} \left[ 0 - 2 \int_{0}^{l} x^2dx \right]$$

$$= -\frac{2}{l} \left[ \frac{x^3}{3} \right]_0^l$$

$$= -\frac{2}{3l} [l^3]$$

$$= -\frac{2l^2}{3}$$

**To find $a_n$**

$$\begin{aligned}
a_n &= \frac{1}{1}\int_{-1}^{1} f(x)\cos\frac{n\pi x}{l}dx \\
&= \frac{1}{l}\int_{-l}^{l}(x-x^2)\cos\frac{n\pi x}{l}dx \\
&= \frac{1}{l}\left[\int_{-l}^{l} x\cos\frac{n\pi x}{l}dx - \int_{-l}^{l} x^2\cos\frac{n\pi x}{l}dx\right] \\
&= \frac{1}{l}\left[0 - 2\int_{0}^{l} x^2\cos\frac{n\pi x}{l}dx\right] \\
&= \frac{-2}{l}\left[x^2\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}} - (2x)\frac{(-\cos\frac{n\pi x}{l})}{\left(\frac{n\pi}{l}\right)^2} + 2\frac{(-\sin\frac{n\pi x}{l})}{\left(\frac{n\pi}{l}\right)^3}\right]_0^l \\
&= \frac{-2}{l}\left[x^2\frac{l}{n\pi}\sin\frac{n\pi x}{l} + 2x\left(\frac{l}{n\pi}\right)^2\cos\frac{n\pi x}{l} - 2\left(\frac{l}{n\pi}\right)^3\sin\frac{n\pi x}{l}\right]_0^l \\
&= -\frac{2}{l}\left[0 + 2l\frac{l^2}{n^2\pi^2}(-1)^n - 0 - \{0+0-0\}\right] \\
&= -\frac{2}{l}\left[\frac{2l^3}{n^2\pi^2}(-1)^n\right] \\
&= -\frac{4l^2}{n^2\pi^2}(-1)^n
\end{aligned}$$

**To find $b_n$**

$$\begin{aligned}
b_n &= \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx \\
&= \frac{1}{l}\int_{-l}^{l}(x-x^2)\sin\frac{n\pi x}{l}dx \\
&= \frac{1}{l}\left[\int_{-l}^{l} x\sin\frac{n\pi x}{l}dx - \int_{-l}^{l} x^2\sin\frac{n\pi x}{l}dx\right] \\
&= \frac{1}{l}\left[2\int_{0}^{l} x\sin\frac{n\pi x}{l}dx + 0\right]
\end{aligned}$$

$$= \frac{2}{l}\left[x\frac{\left(-\cos\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}} - (1)\frac{\left(-\sin\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right]_0^l$$

$$= \frac{2}{l}\left[-x\frac{l}{n\pi}\cos\frac{n\pi x}{l} + \left(\frac{l}{n\pi}\right)^2\sin\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{2}{l}\left[-l\frac{l}{n\pi}(-1)^n - 0 - \{0-0\}\right]$$

$$= -\frac{2l}{n\pi}(-1)^n$$

Substituting in (1),

$$\Rightarrow f(x) = -\frac{2l^2}{3\times 2} + \sum_{n=1}^{\infty}\frac{-4l^2}{n^2\pi^2}(-1)^n\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty}\frac{-2l(-1)^n}{n\pi}\sin\frac{n\pi x}{l}$$

$$= -\frac{l^2}{3} - \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{l} - \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{l}$$

28. Find the Fourier series of $f(x) = e^{-x}$ in $(-1, 1)$.

**Solution:**

Let $2l = 2 \Rightarrow l = 1$

The Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}\ldots\ldots\ldots\ldots\ldots(1)$$

**To find** $a_0$

$$a_0 = \frac{1}{l}\int_{-1}^{1} f(x)dx$$

$$= \int_{-1}^{1} e^{-x}dx$$

$$= \left[\frac{e^{-x}}{-1}\right]_{-1}^{1}$$

$$= -\left[e^{-1} - e^{1}\right]$$

$$= \left[e^{1} - e^{-1}\right]$$

$$= 2\sinh 1$$

**To find $a_n$**

$$\begin{aligned}
a_n &= \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx \\
&= \int_{-1}^{1} e^{-x}\cos n\pi x dx \\
&= \left[\frac{e^{-x}}{(-1)^2+(n\pi)^2}\left[(-1)\cos n\pi x + n\pi \sin n\pi x\right]\right]_{-1}^{1} \\
&= \frac{1}{1+n^2\pi^2}\left[e^{-1}[-(-1)^n] - \{e^1[-(-1)^n]\}\right] \\
&= \frac{(-1)^n}{1+n^2\pi^2}\left[e^1 - e^1\right] \\
&= \frac{(-1)^n}{1+n^2\pi^2} 2\sinh 1
\end{aligned}$$

**To find $b_n$**

$$\begin{aligned}
b_n &= \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx \\
&= \int_{-1}^{1} e^{-x}\sin n\pi x dx \\
&= \left[\frac{e^{-x}}{(-1)^2+(n\pi)^2}\left[(-1)\sin n\pi x - n\pi \cos n\pi x\right]\right]_{-1}^{1} \\
&= \frac{1}{1+n^2\pi^2}\left[e^{-1}[-n\pi(-1)^n] - \{e^1[-n\pi(-1)^n]\}\right] \\
&= \frac{n\pi(-1)^n}{1+n^2\pi^2}\left[e^1 - e^1\right] \\
&= \frac{n\pi(-1)^n}{1+n^2\pi^2} 2\sinh 1
\end{aligned}$$

Sub in (1)

$$\begin{aligned}
\Rightarrow f(x) &= \frac{2\sinh 1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2\pi^2}2\sinh 1\cos n\pi x + \sum_{n=1}^{\infty}\frac{n\pi(-1)^n}{1+n^2\pi^2}2\sinh 1\sin n\pi x \\
&= \sinh 1 + 2\sinh 1\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2\pi^2}\left[\cos n\pi x + n\pi\sin n\pi x\right]
\end{aligned}$$

29. Find the Fourier series expansion of $f(x) = 1 - x^2$ in the interval $(-1, 1)$

## Half Range

**Formula:**

1. Half Range Cosine Series of f(x) in $(0, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{0}^{\infty} a_n \cos nx$$

where $\quad a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx, \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$

2. Half Range Sine Series of f(x) in $(0, l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{0}^{\infty} a_n \cos \tfrac{n\pi x}{l}$$

where $\quad a_0 = \frac{2}{l}\int_0^{l} f(x)dx, \qquad a_n = \frac{2}{l}\int_0^{l} f(x)\cos \tfrac{n\pi x}{l} dx$

3. Half Range Sine Series of f(x) in $(0, \pi)$ is

$$f(x) = \sum_{0}^{\infty} b_n \sin nx$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$

4. Half Range Sine Series of f(x) in $(0, l)$ is

$$f(x) = \sum_{0}^{\infty} b_n \sin \tfrac{n\pi x}{l}$$

where $b_n = \frac{2}{l}\int_0^{l} f(x)\sin \tfrac{n\pi x}{l} dx$

30. Find the Fourier series expansion of $f(x) = x(\pi - x)$ over the interval $(0, \pi)$ as a Fourier cosine series of period $'\pi'$.

Deduce $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots = \frac{\pi^4}{90}$.

**Solution:**

The Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \ldots\ldots\ldots\ldots\ldots.(1)$$

**To find $a_0$**

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^{\pi} f(x)dx \\ &= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)dx \end{aligned}$$

$$\begin{aligned}
&= \frac{2}{\pi}\left[\pi\frac{x^2}{2} - \frac{x^3}{3}\right]_0^\pi \\
&= \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3}\right] \\
&= \frac{2}{\pi}\left[\frac{\pi^3}{6}\right] \\
&= \frac{\pi^2}{3}
\end{aligned}$$

**To find $a_n$**

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^\pi f(x)\cos nx dx \\
&= \frac{2}{\pi}\int_0^\pi (\pi x - x^2)\cos nx dx \\
&= \frac{2}{\pi}\left[(\pi x - x^2)\frac{\sin nx}{n} - (\pi - 2x)\frac{(-\cos nx)}{n^2} + (-2)\frac{(-\sin nx)}{n^3}\right]_0^\pi \\
&= \frac{2}{\pi}\left[(\pi x - x^2)\frac{\sin nx}{n} + (\pi - 2x)\frac{\cos nx}{n^2} + 2\frac{\sin nx}{n^3}\right]_0^\pi \\
&= \frac{2}{\pi}\left[0 + (-\pi)\frac{(-1)^n}{n^2} + 0 - \left\{0 + \pi\frac{1}{n^2} + 0\right\}\right] \\
&= \frac{2}{\pi} \times \frac{-\pi}{n^2}[(-1)^n + 1] \\
&= \begin{cases} 0, & \text{when 'n' is odd;} \\ \frac{-4}{n^2}, & \text{when 'n' is even.} \end{cases}
\end{aligned}$$

Substituting in (1)

$$\begin{aligned}
\Rightarrow f(x) &= \frac{\frac{\pi^2}{3}}{2} + \sum_{n=even}^{\infty} \frac{-4}{n^2}\cos nx \\
&= \frac{\pi^2}{6} - 4\sum_{n=2,4,6,\ldots}^{\infty} \frac{1}{n^2}\cos nx
\end{aligned}$$

Using Parseval's identity

$$\begin{aligned}
\frac{1}{\pi}\int_0^\pi [f(x)]^2\, dx &= \frac{a_0^2}{4} + \frac{1}{2}\sum_0^\infty a_n^2 \\
\frac{1}{\pi}\int_0^\pi [x\pi - x^2]^2\, dx &= \frac{\frac{\pi^4}{9}}{4} + \frac{1}{2}\sum_{n=2,4,6,\ldots}^{\infty} \frac{16}{n^4}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\pi}\int_0^{\pi}[x^2\pi^2 - 2\pi x^3 + x^4]\,dx &= \frac{\pi^4}{9} + 8\left[\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \ldots\right] \\
\frac{1}{\pi}\left[\pi^2\frac{x^3}{3} - 2\pi\frac{x^4}{4} + \frac{x^5}{5}\right]_0^{\pi} &= \frac{\pi^4}{9} + 8\frac{1}{2^4}\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\right] \\
\frac{\pi^5}{\pi}\left[\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right] &= \frac{\pi^4}{9} + \frac{1}{2}\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\right] \\
\pi^4\left[\frac{10-15-6}{30}\right] &= \frac{\pi^4}{9} + \frac{1}{2}\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\right] \\
\pi^4\frac{1}{30} - \frac{\pi^4}{9} &= \frac{1}{2}\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\right] \\
\frac{\pi^4}{180} &= \frac{1}{2}\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\right] \\
\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots &= \frac{\pi^4}{90}
\end{aligned}$$

31. Find the Fourier series expansion of $f(x) = x(l-x)$ over the interval $(0, l)$ as a Fourier cosine series of period 'l'

**Solution:**

The Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

**To find** $a_0$

$$\begin{aligned}
a_0 &= \frac{2}{l}\int_0^{l} f(x)dx \\
&= \frac{2}{l}\int_0^{l}(lx - x^2)dx \\
&= \frac{2}{l}\left[l\frac{x^2}{2} - \frac{(x)^3}{3}\right]_0^{l} \\
&= \frac{2}{l}\left[\frac{l^3}{2} - \frac{l^3}{3}\right] \\
&= \frac{2}{l}\left[\frac{l^3}{6}\right] \\
&= \frac{l^2}{3}
\end{aligned}$$

**To find $a_n$**

$$
\begin{aligned}
a_n &= \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx \\
&= \frac{2}{l}\int_0^l (lx-x^2)\cos\frac{n\pi x}{l}dx \\
&= \frac{2}{l}\left[(lx-x^2)\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}} - (l-2x)\frac{(-\cos\frac{n\pi x}{l})}{\left(\frac{n\pi}{l}\right)^2} + (-2)\frac{(-\sin\frac{n\pi x}{l})}{\left(\frac{n\pi}{l}\right)^3}\right]_0^l \\
&= \frac{2}{l}\left[(lx-x^2)\frac{l}{n\pi}\sin\frac{n\pi x}{l} + (l-2x)\left(\frac{l}{n\pi}\right)^2\cos\frac{n\pi x}{l} + 2\left(\frac{l}{n\pi}\right)^3\sin\frac{n\pi x}{l}\right]_0^l \\
&= \frac{2}{l}\left[-l\frac{l^2}{n^2\pi^2}(-1)^n - \left\{0 + l\frac{l^2}{n^2\pi^2} + 0\right\}\right] \\
&= -\frac{2}{l}\frac{l^3}{n^2\pi^2}[(-1)^n+1] \\
&= -\frac{2l^2}{n^2\pi^2}[(-1)^n+1] \\
&= \begin{cases} 0, & \text{when 'n' is odd;} \\ \frac{-4l^2}{n^2\pi^2}, & \text{when 'n' is even.} \end{cases}
\end{aligned}
$$

Substituting in (1)

$$
\begin{aligned}
\Rightarrow f(x) &= \frac{l^2}{3\times 2} + \sum_{n=even}^{\infty}\frac{-4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l} \\
&= \frac{l^2}{6} - \frac{4l^2}{\pi^2}\sum_{n=2,4,6,\ldots}^{\infty}\frac{\cos\frac{n\pi x}{l}}{n^2}
\end{aligned}
$$

32. Find the half range Fourier cosine series of $f(x) = (\pi - x)^2$ in $(0,\pi)$.Hence find the sum of the series $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots$

**Solution:**

The half range Fourier cosine series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx \ldots\ldots\ldots\ldots\ldots..(1)$$

**To find $a_0$**

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^\pi f(x)dx \\ &= \frac{2}{\pi}\int_0^\pi (\pi-x)^2 dx \\ &= \frac{2}{\pi}\left[\frac{(\pi-x)^3}{3(-1)}\right]_0^\pi \\ &= \frac{2}{-3\pi}[0-\pi^3] \\ &= \frac{2\pi^2}{3} \end{aligned}$$

**To find $a_n$**

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi f(x)\cos nx dx \\ &= \frac{2}{\pi}\int_0^\pi (\pi-x)^2 \cos nx dx \\ &= \frac{2}{\pi}\left[(\pi-x)^2\frac{\sin nx}{n} - 2(\pi-x)(-1)\frac{(-\cos nx)}{n^2} + 2(-1)(-1)\frac{(-\sin nx)}{n^3}\right]_0^\pi \\ &= \frac{2}{\pi}\left[(\pi-x)^2\frac{\sin nx}{n} - 2(\pi-x)\frac{\cos nx}{n^2} - 2\frac{\sin nx}{n^3}\right]_0^\pi \\ &= \frac{2}{\pi}\left[0-0-0-\left\{0-2\pi\frac{1}{n^2}-0\right\}\right] \\ &= \frac{2}{\pi}\left[\frac{2\pi}{n^2}\right] \\ &= \frac{4}{n^2} \end{aligned}$$

Substituting in (1)

$$\Rightarrow f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx$$

By parseval's identity,

$$\frac{1}{\pi}\int_0^\pi [f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2$$

$$\begin{aligned}
\frac{1}{\pi}\int_0^{\pi}(\pi-x)^4dx &= \frac{\frac{4\pi^4}{9}}{4}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4}\\
\frac{1}{\pi}\left[\frac{(\pi-x)^5}{-5}\right]_0^{\pi} &= \frac{4\pi^4}{9}+8\sum_{n=1}^{\infty}\frac{1}{n^4}\\
-\frac{1}{5\pi}\left[0-\pi^5\right] &= \frac{\pi^4}{9}+8\sum_{n=1}^{\infty}\frac{1}{n^4}\\
\frac{1}{\pi}\frac{\pi^5}{5} &= \frac{\pi^4}{9}+8\sum_{n=1}^{\infty}\frac{1}{n^4}\\
8\sum_{n=1}^{\infty}\frac{1}{n^4} &= \frac{\pi^4}{5}-\frac{\pi^4}{9}\\
8\sum_{n=1}^{\infty}\frac{1}{n^4} &= \frac{4\pi^4}{45}\\
\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\dots &= \frac{\pi^4}{90}
\end{aligned}$$

33. Find the half range Fourier cosine series of $f(x)=(x-2)^2$ in $[0,2]$.Hence find the sum of the series $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots$

**Solution:**

Here $l=2$

The half range Fourier cosine series of $f(x)$ is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}\text{.................}(1)$$

**To find $a_0$**

$$\begin{aligned}
a_0 &= \frac{2}{l}\int_0^{l}f(x)dx\\
&= \frac{2}{2}\int_0^{2}(x-2)^2dx\\
&= \left[\frac{(x-2)^3}{3}\right]_0^2\\
&= \frac{1}{3}[0-(-8)]\\
&= \frac{8}{3}
\end{aligned}$$

**To find $a_n$**

$$\begin{aligned}
a_n &= \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx \\
&= \frac{2}{2}\int_0^2 (x-2)^2\cos\frac{n\pi x}{2}dx \\
&= \left[(x-2)^2\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} - 2(x-2)\frac{\left(-\cos\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2} + 2\frac{\left(-\sin\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^3}\right]_0^2 \\
&= \left[(x-2)^2\frac{2}{n\pi}\sin\frac{n\pi x}{2} + 2(x-2)\left(\frac{2}{n\pi}\right)^2\cos\frac{n\pi x}{2} - 2\left(\frac{2}{n\pi}\right)^3\sin\frac{n\pi x}{2}\right]_0^2 \\
&= \left[0+0-0-\left\{-\frac{16}{n^2\pi^2}\right\}\right] \\
&= \frac{16}{n^2\pi^2}
\end{aligned}$$

Substituting in (1)

$$\Rightarrow f(x) = \frac{\frac{8}{3}}{2} + \sum_{n=1}^{\infty}\frac{16}{n^2\pi^2}\cos\frac{n\pi x}{2}$$

$$\Rightarrow f(x) = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{2}.....(2)$$

**Deduction:**

Here $x = 0$ is a point of continuity.

$$\begin{aligned}
f(0) &= \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} \\
4 - \frac{4}{3} &= \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} \\
\frac{8}{3} &= \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} \\
\frac{\pi^2}{6} &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots .....(3)
\end{aligned}$$

Here $x = 2$ is a point of continuity.

$$\begin{aligned}
f(2) &= \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} \\
0 - \frac{4}{3} &= \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}
\end{aligned}$$

$$-\frac{4}{3} = \frac{16}{\pi^2}\left[\frac{(-1)}{1^2} + \frac{1}{2^2} + \frac{(-1)}{3^2} + \dots\right]$$
$$-\frac{4}{3}\frac{\pi^2}{16} = -\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots\right]$$
$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots\dots(4)$$

(3) + (4)

$$2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right] = \frac{\pi^2}{6} + \frac{\pi^2}{12}$$
$$= \frac{3\pi^2}{12}$$
$$= \frac{\pi^2}{4}$$
$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

34. Prove that, if $0 < x < l, x = \frac{l}{2} - \frac{4l}{\pi^2}\left[\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \dots\right]$.

Also deduce that $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots = \frac{\pi^4}{96}$

**Solution:**

The half range Fourier cosine series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}\dots\dots\dots\dots(1)$$

**To find** $a_0$

$$a_0 = \frac{2}{l}\int_0^l f(x)dx$$
$$= \frac{2}{l}\int_0^l xdx$$
$$= \frac{2}{l}\left[\frac{x^2}{2}\right]_0^l$$
$$= \frac{2}{l}\frac{l^2}{2}$$
$$= l$$

**To find $a_n$**

$$\begin{aligned}
a_n &= \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx \\
&= \frac{2}{l}\int_0^l x\cos\frac{n\pi x}{l}dx \\
&= \frac{2}{l}\left[x\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}} - (1)\frac{\left(-\cos\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right]_0^l \\
&= \frac{2}{l}\left[x\frac{l}{n\pi}\sin\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right]_0^l \\
&= \frac{2}{l}\left[0 + \frac{l^2}{n^2\pi^2}(-1)^n - \left\{0 + \frac{l^2}{n^2\pi^2}(1)\right\}\right] \\
&= \frac{2}{l}\frac{l^2}{n^2\pi^2}\left[(-1)^n - 1\right] \\
&= \begin{cases} 0, & \text{when 'n' is even;} \\ \frac{-4l}{n^2\pi^2}, & \text{when 'n' is odd.} \end{cases}
\end{aligned}$$

Sub in (1)

$$\begin{aligned}
\Rightarrow f(x) &= \frac{l}{2} + \sum_{n=1,3,5,\ldots}^{\infty} \frac{-4l}{n^2\pi^2}\cos\frac{n\pi x}{l} \\
&= \frac{l}{2} - \frac{4l}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l} \\
x &= \frac{l}{2} - \frac{4l}{\pi^2}\left[\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \ldots\right]
\end{aligned}$$

Using Parseval's identity

$$\begin{aligned}
\frac{1}{l}\int_0^l [f(x)]^2\,dx &= \frac{a_0^2}{4} + \frac{1}{2}\sum_0^\infty a_n^2 \\
\frac{1}{l}\int_0^l [x]^2\,dx &= \frac{l^2}{4} + \frac{1}{2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{16l^2}{n^4\pi^4} \\
\frac{1}{l}\left[\frac{x^3}{3}\right]_0^l &= \frac{l^2}{4} + \frac{8l^2}{\pi^4}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^4} \\
\frac{1}{l}\left[\frac{l^3}{3}\right] &= \frac{l^2}{4} + \frac{8l^2}{\pi^4}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^4} \\
\frac{l^2}{3} - \frac{l^2}{4} &= \frac{8l^2}{\pi^4}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^4}
\end{aligned}$$

$$\begin{aligned} \frac{l^2}{12} &= \frac{8l^2}{\pi^4} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^4} \\ \frac{\pi^4}{96} &= \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots \end{aligned}$$

35. Expand $f(x) = \sin x, 0 < x < \pi$ in a Fourier cosine series.

36. Obtain the half range cosine series for $f(x) = x$ in $(0, \pi)$ and show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$

37. Obtain the Fourier Cosine series Expansion of $x \sin x$ in $(0, \pi)$ and hence find the value of $1 + \frac{2}{1 \cdot 3} - \frac{2}{3 \cdot 5} + \frac{2}{5 \cdot 7} - \frac{2}{7 \cdot 9}$

38. Find the half range cosine series for the function f(x)is defined as $f(x) = \begin{cases} x, & \text{for } 0 \leq x < \frac{\pi}{2}; \\ \pi - x, & \text{for } \frac{\pi}{2} \leq x < \pi. \end{cases}$

39. Find the half range sine series for a function $f(x) = x(\pi - x), 0 < x < \pi$. Hence deduce $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^3} = \frac{\pi^3}{32}$ or deduce that the value of $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \ldots$.

**Solution:**

The Half range Fourier sine series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx .................(1)$$

**To find** $b_n$

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx \\ &= \frac{2}{\pi} \int_0^{\pi} (x\pi - x^2) \sin nx dx \\ &= \frac{2}{\pi} \left[ (x\pi - x^2) \frac{(-\cos nx)}{n} - (\pi - 2x) \frac{(-\sin nx)}{n^2} + (-2) \frac{(\cos nx)}{n^3} \right]_0^{\pi} \\ &= \frac{2}{\pi} \left[ -(x\pi - x^2) \frac{\cos nx}{n} + (\pi - 2x) \frac{\sin nx}{n^2} - 2 \frac{\cos nx}{n^3} \right]_0^{\pi} \\ &= \frac{2}{\pi} \left[ 0 + 0 - 2 \frac{(-1)^n}{n^3} - \left\{ 0 - 0 - 2 \frac{1}{n^3} \right\} \right] \\ &= \frac{2}{\pi} \frac{2}{n^3} [1 - (-1)^n] \end{aligned}$$

$$= \begin{cases} 0, & \text{when 'n' is even;} \\ \frac{8}{n^3\pi}, & \text{when 'n' is odd.} \end{cases}$$

Sub in (1)

$$\Rightarrow f(x) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{8}{n^3\pi} \sin nx$$

**Deduction:**

Here $x = \frac{\pi}{2}$ is a point of continuity.

$$\begin{aligned} f\left(\frac{\pi}{2}\right) &= \sum_{n=1,3,5,\ldots}^{\infty} \frac{8}{n^3\pi} \sin \frac{n\pi}{2} \\ \frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) &= \frac{8}{\pi}\left[\frac{1}{1^3}\sin\frac{\pi}{2} + \frac{1}{3^3}\sin\frac{3\pi}{2} + \frac{1}{5^3}\sin\frac{5\pi}{2} + \ldots\right] \\ \frac{\pi^2}{4}\frac{\pi}{8} &= \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \ldots \\ \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \ldots &= \frac{\pi^3}{32} \end{aligned}$$

40. Find the half range sine series for the function f(x)is defined as $f(x) = \begin{cases} kx, & \text{for } 0 < x < \frac{l}{2}; \\ k(l-x), & \text{for } \frac{l}{2} < x < l. \end{cases}$

**Solution:**

The Fourier series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \ldots\ldots\ldots\ldots\ldots(1)$$

**To find $b_n$**

$$\begin{aligned} b_n &= \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx \\ &= \frac{2}{l}\left\{\int_0^{\frac{l}{2}} kx\sin\frac{n\pi x}{2}dx + \int_{\frac{l}{2}}^{l} k(l-x)\sin\frac{n\pi x}{2}dx\right\} \\ &= \frac{2k}{l}\left\{\left[x\frac{(-\cos\frac{n\pi x}{l})}{\frac{n\pi}{l}} - (1)\frac{(-\sin\frac{n\pi x}{l})}{\frac{n^2\pi^2}{l^2}}\right]_0^{\frac{l}{2}} + \left[(l-x)\frac{(-\cos\frac{n\pi x}{l})}{\frac{n\pi}{l}} - (-1)\frac{(-\sin\frac{n\pi x}{l})}{\frac{n^2\pi^2}{l^2}}\right]_{\frac{l}{2}}^{l}\right\} \\ &= \frac{2k}{l}\left\{\left[-x\frac{l}{n\pi}\cos\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_0^{\frac{l}{2}} + \left[-(l-x)\frac{l}{n\pi}\cos\frac{n\pi x}{l} - \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_{\frac{l}{2}}^{l}\right\} \end{aligned}$$

$$\begin{aligned}
&= \frac{2k}{l}\left\{-\frac{l}{2}\frac{l}{n\pi}\cos\left(\frac{n\pi}{2}\right)+\frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)+0-\left[-\frac{l}{2}\frac{l}{n\pi}\cos\left(\frac{n\pi}{2}\right)-\frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right]\right\}\\
&= \frac{2k}{l}\frac{2l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\\
&= \frac{4lk}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)
\end{aligned}$$

Substituting in (1)

$$\begin{aligned}
\Rightarrow f(x) &= \sum_{n=1}^{\infty}\frac{4lk}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sin\frac{n\pi x}{l}\\
&= \frac{4lk}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin\frac{n\pi x}{l}
\end{aligned}$$

41. Find the half range sine series for the function f(x)is defined as $f(x)=\begin{cases} x, & \text{for } 0<x<\frac{l}{2};\\ l-x, & \text{for } \frac{l}{2}<x<l.\end{cases}$

42. Expand $f(x)=\begin{cases}\sin x, & \text{for } 0<x<\frac{\pi}{4};\\ \cos x, & \text{for } \frac{\pi}{4}<x<\frac{\pi}{2}.\end{cases}$ in aa series of sine.

43. Find the half range sine series for the function f(x)is defined as $f(x)=\begin{cases} x-1, & \text{for } 0\le x\le 1;\\ 1-x, & \text{for } 1\le x\le 2.\end{cases}$

**Solution:** Here $l=2$

The Fourier series of $f(x)$ is

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}\text{.................}(1)$$

**To find $b_n$**

$$\begin{aligned}
b_n &= \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx\\
&= \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}dx\\
&= \int_0^1 (x-1)\sin\frac{n\pi x}{2}dx+\int_1^2(1-x)\sin\frac{n\pi x}{2}dx\\
&= \left[(x-1)\frac{\left(-\cos\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}-(1)\frac{\left(-\sin\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}\right]_0^1+\left[(1-x)\frac{\left(-\cos\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}-(-1)\frac{\left(-\sin\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}\right]_1^2
\end{aligned}$$

$$= \left[-(x-1)\frac{2}{n\pi}\cos\frac{n\pi x}{2} + \left(\frac{2}{n\pi}\right)^2 \sin\frac{n\pi x}{2}\right]_0^1 + \left[-(1-x)\frac{2}{n\pi}\cos\frac{n\pi x}{2} - \left(\frac{2}{n\pi}\right)^2 \sin\frac{n\pi x}{2}\right]_1^2$$

$$= \left[0 + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} - \left\{\frac{2}{n\pi} + 0\right\}\right] + \left[\frac{2}{n\pi}(-1)^n - 0 - \left\{0 - \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}\right]$$

$$= \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{2}{n\pi}[(-1)^n - 1]$$

$$= \begin{cases} 0, & \text{when 'n' is even;} \\ \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{4}{n\pi}, & \text{when 'n' is odd.} \end{cases}$$

Substituting in (1)

$$\Rightarrow f(x) = \sum_{n=odd}^{\infty}\left(\frac{8}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{4}{n\pi}\right)\sin\frac{n\pi x}{2}$$

44. check that for $0 < x < l, 1 = \frac{4}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + \dots\right)$

45. find the half range sine series for $f(x) = 1 - x$ in $(0, l)$

46. Show that for $0 < x < l, x = \frac{2l}{\pi}\left(\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \dots\right)$ using root mean square value of x, deduce the value of $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$

## HARMONIC ANALYSIS

The process of finding the Fourier series for a function given by numerical value is known as Harmonic analysis.

Formula: $f(x) = \frac{a_0}{2} + (a_1\cos x + b_1 sinx) + (a_2\cos 2x + b_2 sin2x) + \dots$

| | |
|---|---|
| $a_0 = 2\left[\frac{\sum y}{N}\right]$ | |
| $a_1 = 2\left[\frac{\sum y\cos x}{N}\right]$ | $b_1 = 2\left[\frac{\sum y\sin x}{N}\right]$ |
| $a_2 = 2\left[\frac{\sum y\cos 2x}{N}\right]$ | $b_2 = 2\left[\frac{\sum y\sin 2x}{N}\right]$ |
| $\vdots$ | $\vdots$ |

**Note:1**

First Harmonic : $a_1 \cos x + b_1 sinx$

Second Harmonic : $a_2 \cos 2x + b_2 sin2x$

Third Harmonic :$a_3 \cos 3x + b_3 sin3x$.

**Note:2**

Amplitude of First Harmonic $A_1 = \sqrt{a_1^2 + b_1^2}$

Amplitude of Second Harmonic $A_2 = \sqrt{a_2^2 + b_2^2}$

**Type : Degree**

47. Find the Fourier Series expansion of period $2\pi$ for the function $y = f(x)$ which is defined in $(0, 2\pi)$ b means of the table of values given below. Find the series upto the third harmonic.

| x | 0 | $\frac{\pi}{3}$ | $\frac{2\pi}{3}$ | $\pi$ | $\frac{4\pi}{3}$ | $\frac{5\pi}{3}$ | $2\pi$ |
|---|---|---|---|---|---|---|---|
| $y = f(x)$ | 1.0 | 1.4 | 1.9 | 1.7 | 1.5 | 1.2 | 1.0 |

**Solution:**

Since the first and the last value of y is same omit the last one.

The Fourier Series is given by

$$f(x) = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + (a_3 \cos 3x + b_3 \sin 3x).....(1)$$

| $x$ | $y$ | $y \cos x$ | $y \sin x$ | $y \cos 2x$ | $y \sin 2x$ | $y \cos 3y$ | $y \sin 3x$ |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 0 | 1 | 0 | 1 | 0 |
| $\frac{\pi}{3}$ | 1.4 | 0.7 | 1.212 | $-0.7$ | 1.212 | $-1.4$ | 0 |
| $\frac{2\pi}{3}$ | 1.9 | $-0.95$ | 1.65 | $-0.95$ | $-1.645$ | 1.9 | 0 |
| $\pi$ | 1.7 | $-1.7$ | 0 | 1.7 | 0 | $-1.7$ | 0 |
| $\frac{4\pi}{3}$ | 1.5 | 0.75 | $-1.299$ | $-0.75$ | 1.299 | 1.5 | 0 |
| $\frac{5\pi}{3}$ | 1.2 | 0.6 | $-1.039$ | $-0.6$ | $-1.039$ | $-1.2$ | 0 |
| $\sum$ | 8.7 | $-1.1$ | 0.524 | $-0.3$ | $-0.178$ | 0.1 | 0 |

$$a_0 = 2\left[\frac{\sum y}{N}\right] = 2\left[\frac{8.7}{6}\right], \quad a_0 = 2.9$$

$$a_1 = 2\left[\frac{\sum y \cos x}{N}\right] = 2\left[\frac{-1.1}{6}\right], \quad a_1 = -0.367$$

$$b_1 = 2\left[\frac{\sum y \sin x}{N}\right] = 2\left[\frac{0.5196}{6}\right], \quad b_1 = 0.175$$

$$a_2 = 2\left[\frac{\sum y \cos 2x}{N}\right] = 2\left[\frac{-0.3}{6}\right], \quad a_2 = -0.1$$

$$b_2 = 2\left[\frac{\sum y\sin 2x}{N}\right] = 2\left[\frac{-0.1732}{6}\right]$$
$$b_2 = -0.0593$$

$$a_3 = 2\left[\frac{\sum y\cos 3x}{N}\right] = 2\left[\frac{0.1}{6}\right]$$
$$a_3 = 0.033$$

$$b_3 = 2\left[\frac{\sum y\sin 3x}{N}\right] = 0$$

Sub in (1)

$f(x) = 1.45 + (-0.367\cos x + 0.175\sin x) + (-0.1\cos 2x - 0.0593\sin 2x) + 0.033\cos 3x$

48. Determine the first two harmonics of the Fourier series for the following data:

| x | 0 | $\frac{\pi}{3}$ | $\frac{2\pi}{3}$ | $\pi$ | $\frac{4\pi}{3}$ | $\frac{5\pi}{3}$ |
|---|---|---|---|---|---|---|
| $y = f(x)$ | 1.98 | 1.30 | 1.05 | 1.30 | -0.88 | -0.25 |

Ans: $f(x) = 0.75 + (0.37\cos x + 1.004\sin x) + (0.877\cos 2x - 0.109\sin 2x)$

**Type 2: l**

49. Find the Fourier series as far as the second harmonic to represent the function given in the following data.

| x | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| f(x) | 9 | 18 | 24 | 28 | 26 | 20 |

**Solution:**

Here the length of the interval is 6

$i.e., 2l = 6 \Rightarrow l = 3$

w.k.t The Fourier Series is given by

$f(x) = \frac{a_0}{2} + (a_1\cos\frac{\pi x}{l} + b_1\sin\frac{\pi x}{l}) + (a_2\cos\frac{2\pi x}{l} + b_2\sin\frac{2\pi x}{l})$

$f(x) = \frac{a_0}{2} + (a_1\cos\theta + b_1\sin\theta) + (a_2\cos 2\theta + b_2\sin 2\theta), \theta = \frac{\pi x}{3}$.....(1)

| $x$ | $\theta = \frac{\pi x}{3}$ | $y$ | $y\cos\theta$ | $y\cos 2\theta$ | $y\sin\theta$ | $y\sin 2\theta$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 9 | 9 | 9 | 0 | 9 |
| 1 | 60 | 18 | 9 | $-9$ | 15.588 | 15.588 |
| 2 | 120 | 24 | $-12$ | $-12$ | 20.784 | $-20.784$ |
| 3 | 180 | 28 | $-28$ | 28 | 0 | 0 |
| 4 | 240 | 26 | $-13$ | $-13$ | $-22.516$ | 22.516 |
| 5 | 300 | 20 | 10 | $-10$ | $-17.32$ | $-17.32$ |
| $\sum$ | | 125 | $-25$ | $-7$ | $-3.464$ | $-.01$ |

$$a_0 = 2\left[\frac{\sum y}{N}\right] = 2\left[\frac{125}{6}\right], \quad a_0 = 41.66$$

$$a_1 = 2\left[\frac{\sum y\cos\frac{\pi x}{3}}{N}\right] = 2\left[\frac{-25}{6}\right], \quad a_1 = -8.33$$

$$b_1 = 2\left[\frac{\sum y\sin\frac{\pi x}{3}}{N}\right] = 2\left[\frac{-3.464}{6}\right], \quad b_1 = -1.15$$

$$a_2 = 2\left[\frac{\sum y\cos\frac{2\pi x}{3}}{N}\right] = 2\left[\frac{-7}{6}\right], \quad a_2 = -2.33$$

$$b_2 = 2\left[\frac{\sum y\sin\frac{2\pi x}{3}}{N}\right] = 2\left[\frac{-0.01}{6}\right], \quad b_2 = -0.003$$

Sub in (1)

$$f(x) = \frac{41.67}{2} + (-8.33\cos\theta - 1.15\sin\theta) + (-2.33\cos 2\theta - 0.003\sin 2\theta), \theta = \frac{\pi x}{3}$$

$$f(x) = 20.83 - 8.33\cos\frac{\pi x}{3} - 1.15\sin\frac{\pi x}{3} - 2.33\cos\frac{2\pi x}{3} - 0.003\sin\frac{2\pi x}{3}$$

50. Compute the first harmonic of the Fourier series of f(x)

| x | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| f(x) | 4 | 8 | 15 | 7 | 6 | 2 |

Ans : $f(x) = 20.83 + (-2.83\cos\frac{\pi x}{3} + 4.33\sin\frac{\pi x}{3})$

51. The following table gives the variations of a periodic function over a period T

| x | 0 | $\frac{T}{6}$ | $\frac{T}{3}$ | $\frac{T}{2}$ | $\frac{2T}{3}$ | $\frac{5T}{6}$ | $T$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 1.98 | 1.3 | 1.05 | 1.3 | -0.88 | -0.25 | 1.98 |

Show that $f(x) = 0.75 + 0.37\cos\theta + 1.004\sin\theta$ where $\theta = \frac{2\pi x}{T}$.

**Solution:**

Since the first and the last value of y is same omit the last one.

$$i.e., 2l = T \Rightarrow l = \frac{T}{2}$$

w.k.t The Fourier Series is given by

$$f(x) = \frac{a_0}{2} + (a_1\cos\frac{\pi x}{l} + b_1\sin\frac{\pi x}{l})$$
$$f(x) = \frac{a_0}{2} + (a_1\cos\theta + b_1\sin\theta), \theta = \frac{\pi x}{\frac{T}{2}} = \frac{2\pi x}{T} .....(1)$$

| $x$ | $\theta = \frac{2\pi x}{T}$ | $y$ | $y\cos\theta$ | $y\sin\theta$ |
|---|---|---|---|---|
| 0 | 0 | 1.98 | 1.98 | 0 |
| $\frac{T}{6}$ | $\frac{\pi}{3}$ | 1.30 | 0.65 | 1.1258 |
| $\frac{T}{3}$ | $\frac{2\pi}{3}$ | 1.05 | $-0.525$ | 0.9093 |
| $\frac{T}{2}$ | $\pi$ | 1.30 | $-1.30$ | 0 |
| $\frac{2T}{3}$ | $\frac{4\pi}{3}$ | $-0.88$ | 0.44 | 0.762 |
| $\frac{5T}{6}$ | $\frac{5\pi}{3}$ | $-0.25$ | $-0.125$ | 0.2165 |
| $\sum$ | | 4.6 | 1.12 | 3.013 |

$$a_0 = 2\left[\frac{\sum y}{N}\right] = 2\left[\frac{4.6}{6}\right], \quad a_0 = 1.5$$

$$a_1 = 2\left[\frac{\sum y\cos\theta}{N}\right] = 2\left[\frac{1.12}{6}\right], \quad a_1 = 0.37$$

$$b_1 = 2\left[\frac{\sum y\sin\theta}{N}\right] = 2\left[\frac{3.013}{6}\right], \quad b_1 = 1.004$$

Sub in (1)

$f(x) = 0.75 + 0.37\cos\theta + 1.004\sin\theta$

## Complex Form of Fourier Series

52. Find the complex form of the Fourier series of $f(x) = e^{ax}(-\pi < x < \pi)$ in the form.
$e^{ax} = \frac{\sin ha\pi}{\pi}\sum_{-\infty}^{\infty}(-1)^n\frac{a+in}{a^2+n^2}e^{inx}$. And hence prove that $\frac{\pi}{a\sin ha\pi} = \sum_{-\infty}^{\infty}\frac{(-1)^n}{a^2+n^2}$

Solution:

The complex form of Fourier series is

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx} ......(1)$$

**To find $c_n$**

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}e^{-inx}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(a-in)x}dx \\
&= \frac{1}{2\pi}\left[\frac{e^{(a-in)x}}{a-in}\right]_{-\pi}^{\pi} \\
&= \frac{1}{2\pi(a-in)}\left[e^{(a-in)\pi} - e^{-(a-in)\pi}\right] \\
&= \frac{1}{2\pi(a-in)}\frac{a+in}{a+in}\left[e^{a\pi}e^{-in\pi} - e^{-a\pi}e^{in\pi}\right] \\
&= \frac{a+in}{2\pi[a^2-(in)^2]}\left[e^{a\pi}(-1)^n - e^{-a\pi}(-1)^n\right] \\
&= \frac{(a+in)(-1)}{2\pi(a^2+n^2)}\left[e^{a\pi} - e^{-a\pi}\right] \\
&= \frac{(a+in)(-1)^n}{2\pi(a^2+n^2)} 2\sin ha\pi \\
&= \frac{a+in}{\pi(a^2+n^2)}(-1)^n \sin ha\pi
\end{aligned}$$

Sub in (1)

$$\begin{aligned}
f(x) &= \frac{\sin ha\pi}{\pi}\sum_{-\infty}^{\infty}(-1)^n \frac{a+in}{a^2+n^2}e^{inx} \\
e^{ax} &= \frac{\sin ha\pi}{\pi}\sum_{-\infty}^{\infty}(-1)^n \frac{a+in}{a^2+n^2}e^{inx}...(2)
\end{aligned}$$

put $x = 0$

$$\begin{aligned}
f(0) &= \frac{\sin ha\pi}{\pi}\sum_{-\infty}^{\infty}(-1)^n \frac{a+in}{a^2+n^2} \\
1 &= \frac{\sin ha\pi}{\pi}\sum_{-\infty}^{\infty}(-1)^n \frac{a+in}{a^2+n^2} \\
\frac{\pi}{\sin ha\pi} &= \sum_{-\infty}^{\infty}(-1)^n \frac{a+in}{a^2+n^2}
\end{aligned}$$

Equating the real part

$$\frac{\pi}{\sin ha\pi} = \sum_{-\infty}^{\infty}(-1)^n \frac{a}{a^2+n^2}$$

$$\frac{\pi}{a \sin h a\pi} = \sum_{-\infty}^{\infty} \frac{(-1)^n}{a^2+n^2}$$

53. Find the complex form of the Fourier series $e^{-x}$ in $-1 < x < 1$ and hence prove that $\frac{1}{\sin h1} = \sum_{-\infty}^{\infty} (-1)^n \frac{1-in\pi}{1+n^2\pi^2}$.

**Solution:**

Here $2l = 2 \Rightarrow l = 1$

The complex form of Fourier series is

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}, ......(1)$$

**To find** $c_n$

$$\begin{aligned}
c_n &= \frac{1}{2l}\int_{-l}^{l} f(x) e^{\frac{-in\pi x}{l}} dx \\
&= \frac{1}{2}\int_{-1}^{1} e^{-x} e^{-in\pi x} dx \\
&= \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x} dx \\
&= \frac{1}{2}\left[\frac{e^{-(1+in\pi)x}}{-(1+in\pi)}\right]_{-1}^{1} \\
&= \frac{-1}{2(1+in\pi)}\left[e^{-(1+in\pi)} - e^{(1+in\pi)}\right] \\
&= \frac{-1}{2(1+in\pi)} \frac{1-in\pi}{1-in\pi}\left[e^{-1}e^{-in\pi} - e^{1}e^{in\pi}\right] \\
&= \frac{-(1-in\pi)}{2(1^2-(in\pi)^2)}\left[e^{-1}(-1)^n - e^{1}(-1)^n\right] \\
&= \frac{(1-in\pi)}{2(1+n^2\pi^2)}(-1)^n\left[e^{1} - e^{-1}\right] \\
&= \frac{(1-in\pi)}{2(1+n^2\pi^2)}(-1)^n 2\sin h1 \\
&= \frac{1-in\pi}{1+n^2\pi^2}\sin h1(-1)^n
\end{aligned}$$

Sub in (1)

$$\begin{aligned}
f(x) &= \sin h1 \sum_{-\infty}^{\infty}(-1)^n \frac{1-in\pi}{1+n^2\pi^2} e^{in\pi x} \\
e^{-x} &= \sin h1 \sum_{-\infty}^{\infty}(-1)^n \frac{1-in\pi}{1+n^2\pi^2} e^{in\pi x} .....(2)
\end{aligned}$$

$$\begin{aligned} \text{put } x &= 0 \\ f(0) &= \sin h1 \sum_{-\infty}^{\infty} (-1)^n \frac{1-in\pi}{1+n^2\pi^2} \\ 1 &= \sin h1 \sum_{-\infty}^{\infty} (-1)^n \frac{1-in\pi}{1+n^2\pi^2} \\ \sum_{-\infty}^{\infty} (-1)^n \frac{1-in\pi}{1+n^2\pi^2} &= \frac{1}{\sin h1} \end{aligned}$$

## Two Marks

1. Find the Sum of the Fourier series for $f(x) = \begin{cases} x, & \text{for } 0 < x < 1; \\ 2, & \text{for } 1 < x < 2. \end{cases}$ at $x = 1$

   Solution:

   Here $x = 1$ is a point of discontinuity(mid point)

$$\begin{aligned} \text{Sum of the Fourier series} &= \frac{f(1-)+f(1+)}{2} \\ &= \frac{1+2}{2} \\ &= \frac{3}{2} \end{aligned}$$

2. Find the Sum of the Fourier series for $f(x) = x + x^2$ in $-\pi < x < \pi$ at $x = \pi$

   Solution:

   Here $x = \pi$ is a point of discontinuity(end point)

$$\begin{aligned} \text{Sum of the Fourier series} &= \frac{f(-\pi)+f(\pi)}{2} \\ &= \frac{-\pi+\pi^2+\pi+\pi^2}{2} \\ &= \frac{2\pi^2}{2} \\ &= \pi^2 \end{aligned}$$

3. Find the Constant term in the expansion of $\cos^2 x$ as a Fourier Series in the interval $(-\pi, \pi)$.

   Solution:

   Since$f(x)$ is an even function,$b_n = 0$

The Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx................(1)$$

**To find** $a_0$

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2 dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} \frac{1+\cos 2x}{2}dx \\ &= \frac{1}{\pi}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi} dx \\ &= \frac{1}{\pi}(\pi) \\ &= 1 \\ \therefore \text{ the constant term} &= \frac{a_0}{2} = \frac{1}{2} \end{aligned}$$

4. write down $a_0, a_n$ in the expansion of $x + x^3$ as Fourier series in $(-\pi, \pi)$.

   Solution:

$$\begin{aligned} f(x) &= x + x^3 \\ f(-x) &= (-x) + (-x)^3 \\ &= -x - x^3 \\ &= -(x + x^3) \\ f(-x) &= f(x) \end{aligned}$$

$\therefore f(x)$ is an odd function,$\Rightarrow a_0 = a_n = 0$.

5. If the Fourier Series of the function $f(x) = x + x^2$ in the interval $-\pi < x < \pi$ is $\frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{4}{n^2}\cos nx - \frac{2}{n}\sin nx\right]$ then find the value of the Series $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$
   Solution:

Given $f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{4}{n^2}\cos nx - \frac{2}{n}\sin nx\right]$

Put $x = \pi$ is a point of discontinuity(end point)

$$\begin{aligned}
\frac{f(-\pi)+f(\pi)}{2} &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{4}{n^2}(-1)^n - 0\right] \\
\frac{-\pi+\pi^2+\pi+\pi^2}{2} &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^{2n} \\
\frac{2\pi^2}{2} - \frac{\pi^2}{3} &= 4\sum_{n=1}^{\infty}\frac{1}{n^2} \\
\frac{2\pi^2}{3} &= 4\sum_{n=1}^{\infty}\frac{1}{n^2} \\
\frac{2\pi^2}{12} &= \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots \\
\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots &= \frac{\pi^2}{6}
\end{aligned}$$

6. Define Root Mean Square value of a function f(x) over the interval $(a, b)$.

   Solution:

   The Root Mean Square value of a function f(x) over the interval $(a, b)$ is

$$\bar{y} = \sqrt{\frac{\int_a^b [f(x)]^2 dx}{b-a}}$$

(or)

$$\bar{y}^2 = \frac{1}{b-a}\int_a^b [f(x)]^2 dx$$

7. Find the Root mean square value of $f(x) = \pi - x$ in $0 < x < 2\pi$.

   Solution:

$$\begin{aligned}
\bar{y}^2 &= \frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2 dx \\
&= \frac{1}{2\pi}\int_0^{2\pi}[\pi - x]^2 dx \\
&= \frac{1}{2\pi}\left[\frac{(\pi-x)^3}{-3}\right]_0^{2\pi} \\
&= \frac{1}{2\pi}\left[\frac{(-\pi)^3-\pi^3}{-3}\right] \\
&= \frac{1}{2\pi}\left(\frac{-2\pi^3}{-3}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\left(\frac{2\pi^3}{3}\right) \\
&= \frac{\pi^2}{3} \\
\bar{y} &= \frac{\pi}{\sqrt{3}}
\end{aligned}
$$

8. Find the R.M.S. value of $f(x) = 1 - x$ in $0 < x < 1$.

   Solution:

   The R.M.S value of f(x) in $(0, l)$ is

$$
\begin{aligned}
\bar{y}^2 &= \frac{1}{l}\int_0^l [f(x)]^2 dx \\
&= \int_0^1 [1-x]^2 dx \\
&= \left[\frac{(1-x)^3}{-3}\right]_0^1 \\
&= \left[\frac{0-1}{-3}\right] \\
&= \left(\frac{1}{3}\right) \\
&= \frac{1}{3} \\
\bar{y} &= \frac{1}{\sqrt{3}}
\end{aligned}
$$

# Applications of PDE

## WAVE EQUATION

### WITHOUT VELOCITY

1. A string is stretched and fastened to two points l apart. Motion is started by displacing the string into the form $y = k(lx - x^2)$ and then released it from this position at time $t = 0$. Find the displacement of the point of the string at a distance of x from one end at time t.

   Solution:

   The wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

   The Boundary conditions are:

$$(i) y(0,t) = 0 \qquad \forall t > 0$$
$$(ii) y(l,t) = 0 \qquad \forall t > 0$$
$$(iii) \frac{\partial y(x,0)}{\partial t} = 0 \qquad 0 < x < l$$
$$(iv) y(x,0) = k(lx - x^2) \qquad 0 < x < l$$

   The Suitable solution is

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pat + c_4 \sin pat). \;......(1)$$

   Apply condition (i)

$$y(0,t) = c_1(c_3 \cos pat + c_4 \sin pat) = 0$$
$$\Rightarrow \text{Either } c_1 = 0 \text{ or } c_3 \cos pat + c_4 \sin pat = 0$$

   Since $c_3 \cos pat + c_4 \sin pat \neq 0$

$$\Rightarrow \qquad c_1 = 0$$

   Sub in (1)

$$y(x,t) = c_2 \sin px(c_3 \cos pat + c_4 \sin pat). \;......(2)$$

   Apply condition (ii)

$$y(l,t) = c_2 \sin pl(c_3 \cos pat + c_4 \sin pat) = 0$$
$$\Rightarrow \text{Either } c_2 = 0 \text{ or } \sin pl = 0 \text{ or } c_3 \cos pat + c_4 \sin pat = 0$$

   Since $c_3 \cos pat + c_4 \sin pat \neq 0$ and $c_2 \neq 0$[if $c_2 = 0$ we get a trivial solution]

$$\Rightarrow \qquad \sin pl = 0 \qquad\qquad \text{But } \sin n\pi = 0$$
$$\Rightarrow \qquad pl = n\pi$$
$$\Rightarrow \qquad p = \frac{n\pi}{l}$$

   Sub in (2)

$$y(x,t) = c_2 \sin \frac{n\pi x}{l} \left( c_3 \cos \frac{n\pi at}{l} + c_4 \sin \frac{n\pi at}{l} \right) \;.......(3)$$

Diff par w.r.t 't'

$$\frac{\partial y(x,t)}{\partial t} = c_2 \sin\frac{n\pi x}{l}\left[-c_3 \sin\frac{n\pi at}{l}\left(\frac{n\pi a}{l}\right) + c_4 \cos\frac{n\pi at}{l}\left(\frac{n\pi a}{l}\right)\right]$$

Apply condition (iii)

$$\frac{\partial y(x,0)}{\partial t} = c_2 \sin\frac{n\pi x}{l}\left[c_4\left(\frac{n\pi a}{l}\right)\right] = 0$$

$$\Rightarrow \text{Either } c_2 = 0 \text{ or } \sin\frac{n\pi x}{l} = 0 \text{ or } c_4 = 0 \text{ or } \frac{n\pi a}{l} = 0$$

Since $c_2 \neq 0$ , $\sin\frac{n\pi x}{l} \neq 0$ and $\frac{n\pi a}{l} \neq 0$

$$\Rightarrow \quad c_4 = 0$$

Sub in (3)

$$y(x,t) = c_2 \sin\frac{n\pi x}{l} c_3 \cos\frac{n\pi at}{l}$$

The most general form is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}. \;......(4)$$

Apply condition (iv)

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = k(lx - x^2) \qquad 0 < x < l$$

which is a half range Fourier sine series

**To find $c_n$**

$$\begin{aligned}
c_n &= \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx \\
&= \frac{2k}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}dx \\
&= \frac{2k}{l}\left[(lx - x^2)\frac{\left(-\cos\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}} - (l - 2x)\frac{\left(-\sin\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2} + (-2)\frac{(\cos\frac{n\pi x}{l})}{\left(\frac{n\pi}{l}\right)^3}\right]_0^l \\
&= \frac{2k}{l}\left[-2\frac{l^3}{n^3\pi^3}(-1)^n - \left\{-2\frac{l^3}{n^3\pi^3}\right\}\right] \\
&= \frac{2k}{l} \times \frac{2l^3}{n^3\pi^3}[-(-1)^n + 1] \\
&= \frac{4kl^2}{n^3\pi^3}[1 - (-1)^n] \\
&= \begin{cases} \frac{8kl^2}{n^3\pi^3}, & \text{when 'n' is odd;} \\ 0, & \text{when 'n' is even.} \end{cases}
\end{aligned}$$

Substituting in (4)

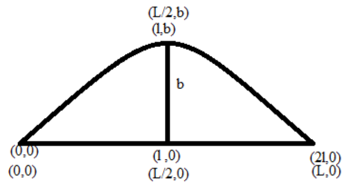
$$y(x,t) = \sum_{n=odd}^{\infty} \frac{8kl^2}{n^3\pi^3}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

$$y(x,t) = \frac{8kl^2}{\pi^3}\sum_{n=1,3,5,...}^{\infty} \frac{1}{n^3}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

2. A string is stretched and fastened to two points l apart. Motion is started by displacing the string into the form $y = 3x(l - x)$ and then released it from this position at time $t = 0$. Find the displacement of the point of the string at a distance of x from one end at time t.

   Ans: Form the Previous question put $k = 3$

3. A tightly stretched string of length 2l is fastened at both ends. The midpoint of the string is displaced by a distance 'b' transversely and the string is released from rest in this position. Find an expression for the transverse displacement of the string at any time during the subsequent motion.




Solution:

$$\text{Let } 2l = L$$

First we find the equation of the String.

Equation between two points$(x_1, y_1)(x_2, y_2)$ is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

Equantion along AB

$$\begin{aligned} \frac{y-0}{b-0} &= \frac{x-0}{\frac{L}{2}-0} \\ y &= b\frac{2}{L}x \\ &= \frac{2bx}{L} \qquad\qquad 0 < x < \frac{L}{2} \end{aligned}$$

Equantion along BC

$$\begin{aligned} \frac{y-b}{0-b} &= \frac{x-\frac{L}{2}}{L-\frac{L}{2}} \\ y - b &= -b\frac{x-\frac{L}{2}}{\frac{L}{2}} \\ y &= b - \frac{2b}{L}\left(x - \frac{L}{2}\right) \\ &= b - \frac{2bx}{L} + b \\ &= 2b - \frac{2bx}{L} \\ &= \frac{2b}{L}(L-x) \qquad\qquad \frac{L}{2} < x < L \end{aligned}$$

$$\therefore y(x,0) = \begin{cases} \frac{2bx}{L}, & \text{for } 0 < x < \frac{L}{2}; \\ \frac{2b}{L}(L-x), & \text{for } \frac{L}{2} < x < L. \end{cases}$$

The wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

The Boundary conditions are:

$$(i) y(0,t) = 0 \qquad \forall t > 0$$

$$(ii) y(L,t) = 0 \qquad \forall t > 0$$

$$(iii) \frac{\partial y(x,0)}{\partial t} = 0 \qquad 0 < x < L$$

$$(iv) y(x,0) = \begin{cases} \frac{2bx}{L}, & \text{for } 0 < x < \frac{L}{2}; \\ \frac{2b}{L}(L-x), & \text{for } \frac{L}{2} < x < L. \end{cases}$$

The Suitable solution is

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pat + c_4 \sin pat). ......(1)$$

Apply condition (i)

$$y(0,t) = c_1(c_3 \cos pat + c_4 \sin pat) = 0$$

$$\Rightarrow \text{Either } c_1 = 0 \text{ or } c_3 \cos pat + c_4 \sin pat = 0$$

Since $c_3 \cos pat + c_4 \sin pat \neq 0$

$$\Rightarrow \qquad c_1 = 0$$

Sub in (1)

$$y(x,t) = c_2 \sin px(c_3 \cos pat + c_4 \sin pat). ......(2)$$

Apply condition (ii)

$$y(L,t) = c_2 \sin pL(c_3 \cos pat + c_4 \sin pat) = 0$$

$$\Rightarrow \text{Either } c_2 = 0 \text{ or } \sin pL = 0 \text{ or } c_3 \cos pat + c_4 \sin pat = 0$$

Since $c_3 \cos pat + c_4 \sin pat \neq 0$ and $c_2 \neq 0$[if $c_2 = 0$ we get a trivial solution]

$$\Rightarrow \qquad \sin pL = 0 \qquad \text{But } \sin n\pi = 0$$

$$\Rightarrow \qquad pL = n\pi$$

$$\Rightarrow \qquad p = \frac{n\pi}{L}$$

Sub in (2)

$$y(x,t) = c_2 \sin \frac{n\pi x}{L}(c_3 \cos \frac{n\pi at}{L} + c_4 \sin \frac{n\pi at}{L}). ......(3)$$

Diff par w.r.t 't'

$$\frac{\partial y(x,t)}{\partial t} = c_2 \sin \frac{n\pi x}{L}\left[-c_3 \sin \frac{n\pi at}{L}\left(\frac{n\pi a}{L}\right) + c_4 \cos \frac{n\pi at}{L}\left(\frac{n\pi a}{L}\right)\right]$$

Apply condition (iii)

$$\frac{\partial y(x,0)}{\partial t} = c_2 \sin \frac{n\pi x}{L}\left[c_4\left(\frac{n\pi a}{L}\right)\right] = 0$$

$$\Rightarrow \text{Either } c_2 = 0 \text{ or } \sin \frac{n\pi x}{L} = 0 \text{ or } c_4 = 0 \text{ or } \frac{n\pi a}{L} = 0$$

Since $c_2 \neq 0$ , $\sin \frac{n\pi x}{L} \neq 0$ and $\frac{n\pi a}{L} \neq 0$

$$\Rightarrow \qquad c_4 = 0$$

Sub in (3)

$$y(x,t) = c_2 \sin \frac{n\pi x}{L} c_3 \cos \frac{n\pi at}{L}$$

The most general form is

$$y(x,t)=\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}. \;......(4)$$

Apply condition (iv)

$$y(x,0)=\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}=k(lx-x^2) \qquad 0<x<l$$

which is a half range Fourier sine series

**To find $c_n$**

$$\begin{aligned}
c_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \\
&= \frac{2}{L}\left\{\int_0^{\frac{L}{2}} \frac{2bx}{L}\sin\frac{n\pi x}{2}dx+\int_{\frac{L}{2}}^{L}\frac{2b}{L}(L-x)\sin\frac{n\pi x}{2}dx\right\} \\
&= \frac{2}{L}\frac{2b}{L}\left\{\left[x\frac{(-\cos\frac{n\pi x}{L})}{\frac{n\pi}{L}}-(1)\frac{(-\sin\frac{n\pi x}{L})}{\frac{n^2\pi^2}{L^2}}\right]_0^{\frac{L}{2}}+\left[(L-x)\frac{(-\cos\frac{n\pi x}{L})}{\frac{n\pi}{L}}-(-1)\frac{(-\sin\frac{n\pi x}{L})}{\frac{n^2\pi^2}{L^2}}\right]_{\frac{L}{2}}^{L}\right\} \\
&= \frac{4b}{L^2}\left\{-\frac{L}{2}\frac{L}{n\pi}\cos\left(\frac{n\pi}{2}\right)+\frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)+0-\left[-\frac{L}{2}\frac{L}{n\pi}\cos\left(\frac{n\pi}{2}\right)-\frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right]\right\} \\
&= \frac{4b}{L^2}\left[\frac{2L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right] \\
&= \frac{8b}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)
\end{aligned}$$

Sub in (4)

$$y(x,t)=\sum_{n=1}^{\infty}\frac{8b}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}$$
$$y(x,t)=\frac{8b}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin\frac{n\pi x}{2l}\cos\frac{n\pi at}{2l}$$

4. A tightly stretched string with fixed end points $x=0$ and $x=l$ initially in a position given by $y(x,0)=y_0\sin^3\left(\frac{\pi x}{l}\right)$. It is released from rest from this position, find the displacement y at any time and at any distance from the end $x=0$. Solution:

The wave equation is

$$\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}$$

The Boundary conditions are:

$$(i)y(0,t)=0 \qquad \forall t>0$$
$$(ii)y(l,t)=0 \qquad \forall t>0$$
$$(iii)\frac{\partial y(x,0)}{\partial t}=0 \qquad 0<x<l$$
$$(iv)y(x,0)=y_0\sin^3\left(\frac{\pi x}{l}\right) \qquad 0<x<l$$

The Suitable solution is

$$y(x,t)=(c_1\cos px+c_2\sin px)(c_3\cos pat+c_4\sin pat). \;......(1)$$

Apply condition (i)

$$y(0,t) = c_1(c_3 \cos pat + c_4 \sin pat) = 0$$

$$\Rightarrow \text{Either } c_1 = 0 \text{ or } c_3 \cos pat + c_4 \sin pat = 0$$

Since $c_3 \cos pat + c_4 \sin pat \neq 0$

$$\Rightarrow \quad c_1 = 0$$

Sub in (1)

$$y(x,t) = c_2 \sin px(c_3 \cos pat + c_4 \sin pat). \;......(2)$$

Apply condition (ii)

$$y(l,t) = c_2 \sin pl(c_3 \cos pat + c_4 \sin pat) = 0$$

$$\Rightarrow \text{Either } c_2 = 0 \text{ or } \sin pl = 0 \text{ or } c_3 \cos pat + c_4 \sin pat = 0$$

Since $c_3 \cos pat + c_4 \sin pat \neq 0$ and $c_2 \neq 0$[if $c_2 = 0$ we get a trivial solution]

$$\Rightarrow \quad \sin pl = 0 \qquad \text{But } \sin n\pi = 0$$

$$\Rightarrow \quad pl = n\pi$$

$$\Rightarrow \quad p = \frac{n\pi}{l}$$

Sub in (2)

$$y(x,t) = c_2 \sin \frac{n\pi x}{l}(c_3 \cos \frac{n\pi at}{l} + c_4 \sin \frac{n\pi at}{l}). \;......(3)$$

Diff par w.r.t 't'

$$\frac{\partial y(x,t)}{\partial t} = c_2 \sin \frac{n\pi x}{l}\left[-c_3 \sin \frac{n\pi at}{l}\left(\frac{n\pi a}{l}\right) + c_4 \cos \frac{n\pi at}{l}\left(\frac{n\pi a}{l}\right)\right]$$

Apply condition (iii)

$$\frac{\partial y(x,0)}{\partial t} = c_2 \sin \frac{n\pi x}{l}\left[c_4\left(\frac{n\pi a}{l}\right)\right] = 0$$

$$\Rightarrow \text{Either } c_2 = 0 \text{ or } \sin \frac{n\pi x}{l} = 0 \text{ or } c_4 = 0 \text{ or } \frac{n\pi a}{l} = 0$$

Since $c_2 \neq 0$ , $\sin \frac{n\pi x}{l} \neq 0$ and $\frac{n\pi a}{l} \neq 0$

$$\Rightarrow \quad c_4 = 0$$

Sub in (3)

$$y(x,t) = c_2 \sin \frac{n\pi x}{l} c_3 \cos \frac{n\pi at}{l}$$

The most general form is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}. \;......(4)$$

Apply condition (iv)

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l} = y_0 \sin^3\left(\frac{\pi x}{l}\right)$$

$$c_1 \sin \frac{\pi x}{l} + c_2 \sin \frac{2\pi x}{l} + c_3 \sin \frac{n\pi x}{l} + \ldots = y_0\left[\frac{3}{4} \sin \frac{\pi x}{l} - \frac{1}{4} \sin \frac{3\pi x}{l}\right]$$

Comparing like coefficients we get,

$$c_1 = \frac{3y_0}{4}, c_2 = 0, c_3 = \frac{-y_0}{4}, c_4 = c_5 = c_6 = \cdots = 0$$

Sub in (1)

$$y(x,0) = c_1 \sin \frac{\pi x}{l} \cos \frac{\pi at}{l} + c_2 \sin \frac{2\pi x}{l} \cos \frac{2\pi at}{l} + c_3 \sin \frac{n\pi x}{l} \cos \frac{3\pi at}{l} + \ldots$$

$$y(x,0) = \frac{3y_0}{4} \sin \frac{\pi x}{l} \cos \frac{\pi at}{l} - \frac{y_0}{4} \sin \frac{n\pi x}{l} \cos \frac{3\pi at}{l}.$$

5. A tightly stretched string with fixed end points $x = 0$ and $x = l$ initially in a position given by $y(x,0) = k \sin\left(\frac{3\pi x}{l}\right) \cos\left(\frac{2\pi x}{l}\right)$. It is released from rest from this position, Determine the displacement $y(x,t)$.

**WITH VELOCITY**

6. A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity $\lambda x(l-x)$,then show that $y(x,t) = \frac{8\lambda l^3}{\pi^4 a}\sum\limits_{n=1,3,5,..}^{\infty}\frac{1}{n^4}\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$.

Solution:

The wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$$

The Boundary conditions are:

$$(i) y(0,t) = 0 \qquad \forall t > 0$$
$$(ii) y(l,t) = 0 \qquad \forall t > 0$$
$$(iii) y(x,0) = 0 \qquad 0 < x < l$$
$$(iv) \frac{\partial y(x,0)}{\partial t} = \lambda x(l-x) \qquad 0 < x < l$$

The Suitable solution is

$$y(x,t) = (c_1\cos px + c_2\sin px)(c_3\cos pat + c_4\sin pat). \;......(1)$$

Apply condition (i)

$$y(0,t) = c_1(c_3\cos pat + c_4\sin pat) = 0$$
$$\Rightarrow \text{Either } c_1 = 0 \text{ or } c_3\cos pat + c_4\sin pat = 0$$

Since $c_3\cos pat + c_4\sin pat \neq 0$

$$\Rightarrow \quad c_1 = 0$$

Sub in (1)

$$y(x,t) = c_2\sin px(c_3\cos pat + c_4\sin pat). \;......(2)$$

Apply condition (ii)

$$y(l,t) = c_2\sin pl(c_3\cos pat + c_4\sin pat) = 0$$
$$\Rightarrow \text{Either } c_2 = 0 \text{ or } \sin pl = 0 \text{ or } c_3\cos pat + c_4\sin pat = 0$$

Since $c_3\cos pat + c_4\sin pat \neq 0$ and $c_2 \neq 0$[if $c_2 = 0$ we get a trivial solution]

$$\Rightarrow \quad \sin pl = 0 \qquad \text{But } \sin n\pi = 0$$
$$\Rightarrow \quad pl = n\pi$$
$$\Rightarrow \quad p = \frac{n\pi}{l}$$

Sub in (2)

$$y(x,t) = c_2\sin\tfrac{n\pi x}{l}(c_3\cos\tfrac{n\pi at}{l} + c_4\sin\tfrac{n\pi at}{l}). \;......(3)$$

Apply condition (iii)

$$y(x,0) = c_2\sin\tfrac{n\pi x}{l}c_3 = 0$$
$$\Rightarrow \text{Either } c_2 = 0 \text{ or } \sin\tfrac{n\pi x}{l} \text{ or } c_3 = 0$$

Since $c_2 = 0 \neq 0$ and $\sin\frac{n\pi x}{l} \neq 0$

$$\Rightarrow \qquad c_3 = 0$$

Sub in (3)

$$y(x,t) = c_2 \sin \tfrac{n\pi x}{l} c_4 \sin \tfrac{n\pi at}{l}$$

The most general form is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin \tfrac{n\pi x}{l} \sin \tfrac{n\pi at}{l}. \;......(4)$$

Diff par w.r.t 't'

$$\frac{\partial y(x,t)}{\partial t} = \sum_{n=1}^{\infty} c_n \sin \tfrac{n\pi x}{l} \cos \tfrac{n\pi at}{l} \left(\tfrac{n\pi a}{l}\right)$$

Apply condition (iv)

$$\frac{\partial y(x,0)}{\partial t} = \sum_{n=1}^{\infty} c_n \sin \tfrac{n\pi x}{l} \left(\tfrac{n\pi a}{l}\right) = \lambda x(l-x)$$

$$\Rightarrow \sum_{n=1}^{\infty} c_n \sin \tfrac{n\pi x}{l} \left(\tfrac{n\pi a}{l}\right) = \sum_{n=1}^{\infty} b_n \sin \tfrac{n\pi x}{l} = \lambda x(l-x) \text{ where } b_n = c_n \tfrac{n\pi a}{l}$$

which is a half range Fourier sine series

**To find $b_n$**

$$\begin{aligned}
b_n &= \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx \\
&= \frac{2\lambda}{l} \int_0^l (lx - x^2) \sin \frac{n\pi x}{l} dx \\
&= \frac{2\lambda}{l} \left[ (lx - x^2) \frac{\left(-\cos \frac{n\pi x}{l}\right)}{\frac{n\pi}{l}} - (l-2x) \frac{\left(-\sin \frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2} + (-2) \frac{\left(\cos \frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3} \right]_0^l \\
&= \frac{2\lambda}{l} \left[ -2\frac{l^3}{n^3\pi^3}(-1)^n - \left\{ -2\frac{l^3}{n^3\pi^3} \right\} \right] \\
&= \frac{2\lambda}{l} \times \frac{2l^3}{n^3\pi^3} [-(-1)^n + 1] \\
c_n \frac{n\pi a}{l} &= \frac{4\lambda l^2}{n^3\pi^3} [1 - (-1)^n] \\
c_n &= \frac{l}{n\pi a} \frac{4\lambda l^2}{n^3\pi^3} [1 - (-1)^n] \\
&= \frac{4\lambda l^3}{n^4\pi^4 a} [1 - (-1)^n] \\
&= \begin{cases} \dfrac{8\lambda l^3}{n^4\pi^4 a}, & \text{when 'n' is odd;} \\ 0, & \text{when 'n' is even.} \end{cases}
\end{aligned}$$

Substituting in (4)

$$y(x,t) = \sum_{n=odd}^{\infty} \frac{8\lambda l^3}{n^4\pi^4 a} \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}$$

$$y(x,t) = \frac{8\lambda l^3}{\pi^4 a} \sum_{n=1,3,5,...}^{\infty} \frac{1}{n^4} \sin \frac{n\pi x}{l} \cos \frac{n\pi at}{l}$$

7. A tightly stretched string of length 'l' is initially at rest in its equilibrium position and each of its point given a velocity $v_0 \sin^3\left(\frac{\pi x}{l}\right)$.Determine the displacement $y(x,t)$. Solution:

   The wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

The Boundary conditions are:

$$(i) y(0,t) = 0 \qquad \forall t > 0$$
$$(ii) y(l,t) = 0 \qquad \forall t > 0$$
$$(iii) y(x,0) = 0 \qquad 0 < x < l$$
$$(iv) \frac{\partial y(x,0)}{\partial t} = \lambda x(l-x) \qquad 0 < x < l$$

The Suitable solution is

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pat + c_4 \sin pat). \text{......}(1)$$

Apply condition (i)

$$y(0,t) = c_1(c_3 \cos pat + c_4 \sin pat) = 0$$
$$\Rightarrow \text{Either } c_1 = 0 \text{ or } c_3 \cos pat + c_4 \sin pat = 0$$

Since $c_3 \cos pat + c_4 \sin pat \neq 0$

$$\Rightarrow \qquad c_1 = 0$$

Sub in (1)

$$y(x,t) = c_2 \sin px(c_3 \cos pat + c_4 \sin pat). \text{......}(2)$$

Apply condition (ii)

$$y(l,t) = c_2 \sin pl(c_3 \cos pat + c_4 \sin pat) = 0$$
$$\Rightarrow \text{Either } c_2 = 0 \text{ or } \sin pl = 0 \text{ or } c_3 \cos pat + c_4 \sin pat = 0$$

Since $c_3 \cos pat + c_4 \sin pat \neq 0$ and $c_2 \neq 0$[if $c_2 = 0$ we get a trivial solution]

$$\Rightarrow \qquad \sin pl = 0 \qquad\qquad \text{But } \sin n\pi = 0$$
$$\Rightarrow \qquad pl = n\pi$$
$$\Rightarrow \qquad p = \frac{n\pi}{l}$$

Sub in (2)

$$y(x,t) = c_2 \sin \tfrac{n\pi x}{l}(c_3 \cos \tfrac{n\pi at}{l} + c_4 \sin \tfrac{n\pi at}{l}). \text{......}(3)$$

Apply condition (iii)

$$y(x,0) = c_2 \sin \tfrac{n\pi x}{l} c_3 = 0$$
$$\Rightarrow \text{Either } c_2 = 0 \text{ or } \sin \tfrac{n\pi x}{l} \text{ or } c_3 = 0$$

Since $c_2 = 0 \neq 0$ and $\sin \frac{n\pi x}{l} \neq 0$

$$\Rightarrow \qquad c_3 = 0$$

Sub in (3)

$$y(x,t) = c_2 \sin \tfrac{n\pi x}{l} c_4 \sin \tfrac{n\pi at}{l}$$

The most general form is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin \tfrac{n\pi x}{l} \sin \tfrac{n\pi at}{l}. \text{......}(4)$$

Diff par w.r.t 't'

$$\frac{\partial y(x,t)}{\partial t} = \sum_{n=1}^{\infty} c_n \sin\tfrac{n\pi x}{l} \cos\tfrac{n\pi at}{l}\left(\tfrac{n\pi a}{l}\right)$$

Apply condition (iv)

$$\frac{\partial y(x,0)}{\partial t} = \sum_{n=1}^{\infty} c_n \sin\tfrac{n\pi x}{l}\left(\tfrac{n\pi a}{l}\right) = v_0 \sin^3\left(\tfrac{\pi x}{l}\right)$$

$$c_1\frac{\pi x}{l}\sin\tfrac{\pi x}{l} + c_2\frac{2\pi x}{l}\sin\tfrac{2\pi x}{l} + c_3\frac{3\pi x}{l}\sin\tfrac{n\pi x}{l} + \ldots = v_0\left[\frac{3}{4}\sin\tfrac{\pi x}{l} - \frac{3}{4}\sin\tfrac{3\pi x}{l}\right]$$

Comparing like coefficients we get,

$$c_1\frac{\pi x}{l} = \frac{3y_0}{4}, c_2 = 0, c_3\frac{3\pi x}{l} = \frac{-y_0}{4}, c_4 = c_5 = c_6 = \cdots = 0$$
$$c_1 = \frac{l}{\pi x}\frac{3y_0}{4}, c_2 = 0, c_3 = \frac{l}{3\pi x}\left(\frac{-y_0}{4}\right), c_4 = c_5 = c_6 = \cdots = 0$$

Sub in (1)

$$y(x,0) = c_1\sin\tfrac{\pi x}{l} + c_2\sin\tfrac{2\pi x}{l} + c_3\sin\tfrac{n\pi x}{l} + \ldots$$
$$y(x,0) = \frac{3v_0 l}{4\pi a}\sin\tfrac{\pi x}{l} - \frac{v_0 l}{12\pi a}\sin\tfrac{n\pi x}{l}.$$

8. A string of length 'l' is initially at rest in its equilibrium position and motion is started by giving each of its points a velocity $V = \begin{cases} cx, & \text{for } 0 < x \le \frac{l}{2}; \\ c(l-x), & \text{for } \frac{l}{2} < x \le l. \end{cases}$ Find the displacement $y(x,t)$.

## 1D HEAT FLOW EQUATION

1. A metal bar 30 cm long has its ends A and B kept at $20°C$ and $80°C$ respectively, until steady state conditions prevail. The temperature at each end is then suddenly reduced to $0°C$ and kept so. Find the resulting temperature distribution function $u(x,t)$ taking $x = 0$ at A.

Solution:

The 1D heat equation is

$$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial^x}$$


When steady state condition prevails, $\frac{\partial u}{\partial t} = 0$

we get,

$$\begin{aligned} u(x) &= \frac{\theta_2 - \theta_1}{l}x + \theta_1 \\ &= \frac{80-20}{30}x + 20 \\ &= 2x + 20 \qquad 0 \le x \le 30 \end{aligned}$$

The Boundary conditions are:

$$(i)u(0,t) = 0 \qquad \forall t > 0$$
$$(ii)u(30,t) = 0 \qquad \forall t > 0$$
$$(iii)u(x,0) = 2x + 20 \qquad 0 \le x \le 30$$

The Suitable solution is

$$u(x,t) = (c_1 \cos px + c_2 \sin px)e^{-\alpha^2 p^2 t}. ......(1)$$

Apply condition (i)

$$u(0,t) = c_1 e^{-\alpha^2 p^2 t} = 0$$

$$\Rightarrow \text{Either } c_1 = 0 \text{ or } e^{-\alpha^2 p^2 t} = 0$$

Since $e^{-\alpha^2 p^2 t} \neq 0$

$$\Rightarrow \qquad c_1 = 0$$

Sub in (1)

$$u(x,t) = c_2 \sin px e^{-\alpha^2 p^2 t}. ......(2)$$

Apply condition (ii)

$$u(30,t) = c_2 \sin 30p e^{-\alpha^2 p^2 t} = 0$$

$$\Rightarrow \text{Either } c_2 = 0 \text{ or } \sin 30p = 0 \text{ or } e^{-\alpha^2 p^2 t} = 0$$

Since $e^{-\alpha^2 p^2 t} \neq 0$ and $c_2 \neq 0$[if $c_2 = 0$ we get a trivial solution]

$$\Rightarrow \qquad \sin 30p = 0 \qquad\qquad \text{But } \sin n\pi = 0$$

$$\Rightarrow \qquad 30p = n\pi$$

$$\Rightarrow \qquad p = \frac{n\pi}{30}$$

Sub in (2)

$$u(x,t) = c_2 \sin \tfrac{n\pi x}{30} e^{-\alpha^2 \left(\frac{n\pi}{30}\right)^2 t}$$

The most general form is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin \tfrac{n\pi x}{30} e^{\frac{-\alpha^2 n^2 \pi^2 t}{900}}. ......(3)$$

Apply condition (iii)

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin \tfrac{n\pi x}{30} = 2x + 20$$

which is a half range Fourier sine series

**To find $c_n$**

$$\begin{aligned}
c_n &= \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l} dx \\
&= \frac{2}{30}\int_0^{30} (2x+20) \sin \frac{n\pi x}{30} dx \\
&= \frac{1}{15}\left[(2x+20)\frac{\left(-\cos \frac{n\pi x}{30}\right)}{\frac{n\pi}{30}} - 2\frac{\left(-\sin \frac{n\pi x}{30}\right)}{\left(\frac{n\pi}{30}\right)^2}\right]_0^{30} \\
&= \frac{1}{15}\left[-80\frac{30}{n\pi}(-1)^n - \left(-20\frac{30}{n\pi}\cdot 1\right)\right] \\
&= \frac{1}{15}\frac{30}{n\pi}\cdot 20\left[-4(-1)^n + 1\right] \\
&= \frac{40}{n\pi}\left[1 - 4(-1)^n\right]
\end{aligned}$$

Substituting in (3)

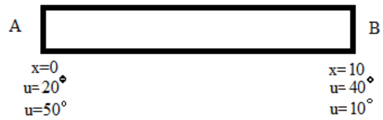
$$u(x,t) = \sum_{n=1}^{\infty} \frac{40}{n\pi}\left[1-4(-1)^n\right]\sin\frac{n\pi x}{30} e^{\frac{-\alpha^2 n^2 \pi^2 t}{900}}$$
$$u(x,t) = \frac{40}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left[1-4(-1)^n\right]\sin\frac{n\pi x}{30} e^{\frac{-\alpha^2 n^2 \pi^2 t}{900}}$$

2. A bar,10 cm long, with insulated sides, has its ends A and B kept at $20°C$ and $40°C$ respectively until steady state condition prevail. The temperature at A is suddenly raised to $50°C$ and at the same instant at B is lowered to $10°C$. Find the subsequent temperature at any point of the bar at any time.

Solution:

The 1D heat equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x}$$




Here there are two steady states,

The solution may be, $u(x,t) = u_s(x) + u_T(x,t)$. .....$(I)$

$$\begin{aligned} u(x,0) &= \frac{\theta_2-\theta_1}{l}x+\theta_1 \\ &= \frac{40-20}{10}x+20 \\ &= 2x+20 \end{aligned} \qquad \begin{aligned} u_s(x) &= \frac{\theta_2-\theta_1}{l}x+\theta_1 \\ &= \frac{10-50}{10}x+50 \\ &= -4x+50 \end{aligned}$$

$(I) \Rightarrow u(x,t) = -4x+50+u_T(x,t)$

The Boundary conditions are:

$$(i) u(0,t) = 50 \qquad \forall t > 0$$
$$(ii) u(10,t) = 10 \qquad \forall t > 0$$
$$(iii) u(x,0) = 2x+20 \qquad 0 \le x \le 10$$

The Suitable solution is

$$u(x,t) = -4x+50+(c_1\cos px + c_2 \sin px)e^{-\alpha^2 p^2 t}. ......(1)$$

Apply condition (i)

$$u(0,t) = 50 + c_1 e^{-\alpha^2 p^2 t} = 50$$
$$\Rightarrow c_1 e^{-\alpha^2 p^2 t} = 0$$
$$\Rightarrow \text{Either } c_1 = 0 \text{ or } e^{-\alpha^2 p^2 t} = 0$$

Since $e^{-\alpha^2 p^2 t} \neq 0$

$$\Rightarrow \qquad c_1 = 0$$

Sub in (1)

$$u(x,t) = -4x+50+c_2 \sin px e^{-\alpha^2 p^2 t}. ......(2)$$

Apply condition (ii)

$$u(10,t) = 10 + c_2 \sin 10pe^{-\alpha^2 p^2 t} = 10$$
$$\Rightarrow c_2 \sin 10pe^{-\alpha^2 p^2 t} = 0$$
$$\Rightarrow \text{Either } c_2 = 0 \text{ or } \sin 10p = 0 \text{ or } e^{-\alpha^2 p^2 t} = 0$$

Since $e^{-\alpha^2 p^2 t} \neq 0$ and $c_2 \neq 0$[if $c_2 = 0$ we get a trivial solution]

$$\Rightarrow \quad \sin 10p = 0 \text{ But } \sin n\pi = 0$$
$$\Rightarrow \quad 10p = n\pi$$
$$\Rightarrow \quad p = \frac{n\pi}{10}$$

Sub in (2)

$$u(x,t) = -4x + 50 + c_2 \sin \tfrac{n\pi x}{10} e^{-\alpha^2 \left(\frac{n\pi}{10}\right)^2 t}$$

The most general form is

$$u(x,t) = -4x + 50 + \sum_{n=1}^{\infty} c_n \sin \tfrac{n\pi x}{10} e^{\frac{-\alpha^2 n^2 \pi^2 t}{100}}. \text{ ......(3)}$$

Apply condition (iii)

$u(x,0) = -4x + 50 + \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{10} = 2x + 20$

$\sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{10} = 6x - 30$

which is a half range Fourier sine series

**To find $c_n$**

$$\begin{aligned}
c_n &= \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx \\
&= \frac{2}{10} \int_0^{10} (6x - 30) \sin \frac{n\pi x}{10} dx \\
&= \frac{1}{5} \left[ (6x - 30) \frac{\left(-\cos \frac{n\pi x}{10}\right)}{\frac{n\pi}{10}} - 6 \frac{\left(-\sin \frac{n\pi x}{10}\right)}{\left(\frac{n\pi}{10}\right)^2} \right]_0^{10} \\
&= \frac{1}{5} \left[ -30 \frac{10}{n\pi} (-1)^n - \left( 30 \frac{10}{n\pi} \cdot 1 \right) \right] \\
&= \frac{1}{15} \frac{30}{n\pi} \cdot 30 \left[ -(-1)^n - 1 \right] \\
&= \frac{-60}{n\pi} \left[ 1 + (-1)^n \right] \\
&= \begin{cases} 0, & \text{when 'n' is odd;} \\ \dfrac{-120}{n\pi}, & \text{when 'n' is even.} \end{cases}
\end{aligned}$$

Substituting in (3)

$$u(x,t) = -4x + 50 + \sum_{n=2,4,6,\cdots}^{\infty} \frac{-120}{n\pi} \sin \tfrac{n\pi x}{10} e^{\frac{-\alpha^2 n^2 \pi^2 t}{100}}$$
$$u(x,t) = -4x + 50 - \frac{120}{\pi} \sum_{n=2,4,6,\cdots}^{\infty} \frac{1}{n} \sin \tfrac{n\pi x}{30} e^{\frac{-\alpha^2 n^2 \pi^2 t}{100}}$$

A metal bar 20 cm long has its ends A and B kept at $30°C$ and $90°C$ respectively, until steady state conditions prevail. The temperature at each end is then suddenly reduced to $0°C$ and kept so. Find the resulting temperature distribution function $u(x,t)$ at a distance x from A at time t.

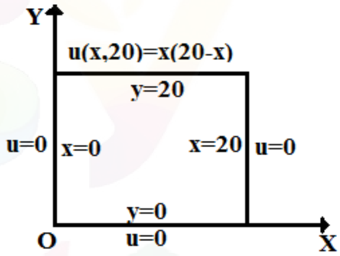
Ans:$u(x,t) = \frac{60}{\pi} \sum_{n=1}^{\infty} \frac{[1-3(-1)^n]}{l} \sin \frac{n\pi x}{20} e^{\frac{-\alpha^2 n^2 \pi^2 t}{400}}$

A bar,l cm long, with insulated sides, has its ends A and B kept at $30°C$ and $80°C$ respectively until steady state condition prevail. The temperature of the end B is suddenly reduced to $60°C$ and that of A increased to $40°C$ . Find the temperature distribution of the rod after time t.

Ans:$u(x,t) = \frac{20}{l}x + 40 - \frac{20}{\pi} \sum_{n=2,4,6,\cdots}^{\infty} \frac{[1+2(-1)^n]}{n} \sin \frac{n\pi x}{l} e^{\frac{-\alpha^2 n^2 \pi^2 t}{l^2}}$

## 2D HEAT FLOW EQUATION

3. A square plate is bounded by the lines $x = 0, y = 0, x = 20$ and $y = 20$. Its faces are insulated. The temperatue along the upper horizontal edge is given by $u(x, 20) = x(20 - x), 0 < x < 20$ while other three edges are kept at $0°C$. Find the steady state temperature in the plate.




Solution:

The 2D heat flow equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The Boundary conditions are:

$$(i) u(0,y) = 0 \qquad 0 \le y \le 20$$

$$(ii) u(20,y) = 0 \qquad 0 \le y \le 20$$

$$(iii) u(x,0) = 0 \qquad 0 \le x \le 20$$

$$(iv) u(x,20) = x(20-x) \qquad 0 \le x \le 20$$

The Suitable solution is

$$u(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}). ......(1)$$

Apply condition (i)

$$u(0,y) = c_1(c_3 e^{py} + c_4 e^{py}) = 0$$

$$\Rightarrow \text{Either } c_1 = 0 \text{ or } c_3 e^{py} + c_4 e^{-py}$$

Since $c_3 e^{py} + c_4 e^{py} \neq 0$

$$\Rightarrow \qquad c_1 = 0$$

Sub in (1)

$$u(x,y) = c_2 \sin px (c_3 e^{py} + c_4 e^{-py}). ......(2)$$

Apply condition (ii)

$$u(20,y) = c_2 \sin 20p (c_3 e^{py} + c_4 e^{-py}) = 0$$

$$\Rightarrow \text{Either } c_2 = 0 \text{ or } \sin 20p = 0 \text{ or } c_3 e^{py} + c_4 e^{-py} = 0$$

Since $c_3e^{py} + c_4e^{-py} \neq 0$ and $c_2 \neq 0$[if $c_2 = 0$ we get a trivial solution]

$$\Rightarrow \quad \sin 20p = 0 \qquad \text{But } \sin n\pi = 0$$

$$\Rightarrow \quad 20p = n\pi$$

$$\Rightarrow \quad p = \frac{n\pi}{20}$$

Sub in (2)

$$u(x,y) = c_2 \sin\frac{n\pi x}{20}\left(c_3e^{\frac{n\pi y}{20}} + c_4e^{\frac{-n\pi y}{20}}\right) \text{.......(3)}$$

Apply condition (iii)

$$u(x,0) = c_2 \sin\frac{n\pi x}{20}(c_3 + c_4) = 0$$

$$\Rightarrow \text{Either } c_2 = 0 \text{ or } \sin\frac{n\pi x}{20} \text{ or } c_3 + c_4 = 0$$

Since $c_2 \neq 0$ and $\sin\frac{n\pi x}{20} \neq 0$

$$\Rightarrow \quad c_3 + c_4 = 0$$

$$\Rightarrow \quad c_4 = -c_3$$

Sub in (3)

$$u(x,y) = c_2 \sin\frac{n\pi x}{20}\left(c_3e^{\frac{n\pi y}{20}} - c_3e^{\frac{-n\pi y}{20}}\right)$$

$$u(x,y) = c_2 \sin\frac{n\pi x}{20}c_3(e^{\frac{n\pi y}{20}} - e^{\frac{-n\pi y}{20}})$$

$$u(x,y) = c_2 \sin\frac{n\pi x}{20}c_3(2\sinh\tfrac{n\pi y}{20})$$

The most general form is

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{20}\sinh\tfrac{n\pi y}{20}. \text{......(4)}$$

Apply condition (iv)

$$u(x,20) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{20}\sinh n\pi = x(20-x) \qquad 0 < x < l$$

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{20} = x(20-x) \qquad \text{where } b_n = c_n \sinh n\pi$$

which is a half range Fourier sine series

**To find $b_n$**

$$\begin{aligned}
b_n &= \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx \\
&= \frac{2}{20}\int_0^{20}(20x - x^2)\sin\frac{n\pi x}{20}dx \\
&= \frac{1}{10}\left[(20x - x^2)\frac{\left(-\cos\frac{n\pi x}{20}\right)}{\frac{n\pi}{20}} - (20-2x)\frac{\left(-\sin\frac{n\pi x}{20}\right)}{\left(\frac{n\pi}{20}\right)^2} + (-2)\frac{(\cos\frac{n\pi x}{20})}{\left(\frac{n\pi}{20}\right)^3}\right]_0^{20} \\
&= \frac{1}{10}\left[-2\frac{20^3}{n^3\pi^3}(-1)^n - \left\{-2\frac{20^3}{n^3\pi^3}\right\}\right] \\
&= \frac{1}{10} \times \frac{2\cdot 20^3}{n^3\pi^3}[-(-1)^n + 1] \\
b_n &= \frac{4\cdot 20^2}{n^3\pi^3}[1 - (-1)^n] \\
c_n \sinh n\pi &= \frac{4\cdot 20^2}{n^3\pi^3}[1 - (-1)^n] \\
c_n &= \frac{1}{\sinh n\pi}\frac{4\cdot 20^2}{n^3\pi^3}[1 - (-1)^n]
\end{aligned}$$

$$= \begin{cases} \frac{8 \cdot 20^2}{n^3\pi^3} \frac{1}{\sinh n\pi}, & \text{when 'n' is odd;} \\ 0, & \text{when 'n' is even.} \end{cases}$$

Substituting in (4)

$$u(x,y) = \sum_{n=odd}^{\infty} \frac{3200}{n^3\pi^3 \sinh n\pi} \sin\frac{n\pi x}{20} \sinh\frac{n\pi y}{20}$$
$$u(x,y) = \frac{3200}{\pi^3} \sum_{n=odd}^{\infty} \frac{1}{n^3 \sinh n\pi} \sin\frac{n\pi x}{20} \sinh\frac{n\pi y}{20}$$

4. A rectangular plate with insulated surfaces is 20 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature of the short edge $x = 0$ is given by $u = \begin{cases} 10y, & 0 \le y \le 10; \\ 10(20-y), & 10 \le y \le 20. \end{cases}$ and the two long edges as well as the other short edge are kept at $0°C$, find the steady state temperature distribution in the plate. Solution:

The 2D heat flow equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The Boundary conditions are:

$$(i) u(x,0) = 0$$
$$(ii) u(x,20) = 0$$
$$(iii) u(\infty,y) = 0 \qquad 0 \le y \le 20$$
$$(iv) u(0,y) = \begin{cases} 10y, & 0 \le y \le 10; \\ 10(20-y), & 10 \le y \le 20. \end{cases}$$

The Suitable solution is

$$u(x,y) = (c_1e^{px} + c_2e^{-px})(c_3\cos py + c_4\sin py). \text{......(1)}$$

Apply condition (i)

$$u(x,0) == (c_1e^{px} + c_2e^{-px})c_3 = 0$$
$$\Rightarrow \text{Either } c_1e^{px} + c_2e^{-px} \text{ or } c_1 = 0$$

Since $c_1e^{px} + c_2e^{-px}$ or $\neq 0$

$$\Rightarrow \qquad c_3 = 0$$

Sub in (1)

$$u(x,y) = (c_1e^{px} + c_2e^{-px})c_4\sin py. \text{......(2)}$$

Apply condition (ii)

$$u(x,20) = (c_1e^{px} + c_2e^{-px})c_4\sin 20p = 0$$
$$\Rightarrow \text{Either } c_1e^{px} + c_2e^{-px} = 0 \text{ or } c_4 = 0 \text{ or } \sin 20p = 0$$

Since $c_1e^{px} + c_2e^{-px} \neq 0$ and $c_4 \neq 0$[if $c_4 = 0$ we get a trivial solution]

$$\Rightarrow \qquad \sin 20p = 0 \qquad\qquad \text{But } \sin n\pi = 0$$

$$\Rightarrow \quad 20p = n\pi$$
$$\Rightarrow \quad p = \frac{n\pi}{20}$$

Sub in (2)

$$u(x,y) = (c_1 e^{\frac{n\pi x}{20}} + c_2 e^{\frac{-n\pi x}{20}}) c_4 \sin\frac{n\pi y}{20}. \ ......(3)$$

Apply condition (iii)

$$u(\infty, y) = c_1 c_4 \sin\frac{n\pi y}{20} = 0$$
$$\Rightarrow \text{Either } c_1 = 0 \text{ or } c_4 = 0 \text{ or } \sin\frac{n\pi y}{20}$$

Since $c_4 \neq 0$ and $\sin\frac{n\pi y}{20} \neq 0$

$$\Rightarrow \quad c_2 = 0 \text{ for if } c_2 \neq 0, \text{then } e^{\lambda x} \to \infty \Rightarrow u \to \infty$$

which is a contradiction for $u = 0$.

Sub in (3)

$$u(x,y) = c_2 e^{\frac{n\pi x}{20}} c_4 \sin\frac{n\pi y}{20}$$

The most general form is

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi y}{20} e^{\frac{n\pi x}{20}}. \ ......(4)$$

Apply condition (iv)

$$u(0,y) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi y}{20} = \begin{cases} 10y, & 0 \le y \le 10; \\ 10(20-y), & 10 \le y \le 20. \end{cases}$$

which is a half range Fourier sine series

**To find $c_n$**

$$\begin{aligned}
c_n &= \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dy \\
&= \frac{2}{20}\left[\int_0^{10} 10y\sin\frac{n\pi y}{20}dy + \int_{10}^{20} 10(20-y)\sin\frac{n\pi y}{20}dy\right] \\
&= \frac{1}{10}10\left[\left\{y\left(\frac{-\cos\frac{n\pi y}{20}}{\frac{n\pi}{20}}\right) - (1)\left(\frac{-\sin\frac{n\pi y}{20}}{\frac{n^2\pi^2}{20^2}}\right)\right\}_0^{10} + \left\{(20-y)\left(\frac{-\cos\frac{n\pi y}{20}}{\frac{n\pi}{20}}\right) - (-1)\left(\frac{-\sin\frac{n\pi y}{20}}{\frac{n^2\pi^2}{20^2}}\right)\right\}_{10}^{20}\right] \\
&= \left[-10\frac{20}{n\pi}\cos\frac{n\pi}{2} + \frac{20^2}{n^2\pi^2}\sin\frac{n\pi}{2} - \left\{-10\frac{20}{n\pi}\cos\frac{n\pi}{2} - \frac{20^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}\right] \\
&= 2\frac{20^2}{n^2\pi^2}\sin\frac{n\pi}{2} \\
c_n &= \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2} \\
&= \begin{cases} \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}, & \text{when 'n' is odd;} \\ 0, & \text{when 'n' is even.} \end{cases}
\end{aligned}$$

Substituting in (4)

$$u(x,y) = \sum_{n=odd}^{\infty} \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2} e^{\frac{n\pi x}{20}} \sin\frac{n\pi y}{20}$$
$$u(x,y) = \frac{3200}{\pi^2}\sum_{n=odd}^{\infty} \frac{1}{n^2} e^{\frac{n\pi x}{20}} \sin\frac{n\pi y}{20}$$

Find the steady state temperature distribution in a rectangular plate of sides a and b insulated at the lateral surfaces and satisfying the boundary conditions

$u(0, y) = 0, u(a, y) = 0, 0 \leq y \leq b$

$u(x, b) = 0, u(x, 0) = x(a - x), 0 \leq x \leq a$

A rectangular plate with insulated surface is 10cm wide and so long compared to its width that it may be considered infinite in length without introducing appreciable error. The temperature at short edge $y = 0$ is given by

$$u = \begin{cases} 20x, & 0 \leq x \leq 5; \\ 20(10 - x), & 5 \leq x \leq 10. \end{cases}$$

and all the other three edges are kept at $0 \circ C$. Find the steady state temperature at any point in the plate.

Ans:$u(x, y) = \frac{800}{\pi^2} \sum_{n=odd}^{\infty} \frac{1}{n^2} e^{\frac{n\pi y}{10}} \sin \frac{n\pi x}{10}$

An infinite long rectangular plate with insulated surface in 10cm wide. The two long edges and one short edge are kept at zero temperature. While the other short edge $x = 0$ is kept at temperature given by

$$u = \begin{cases} 20y, & 0 \leq y \leq 5; \\ 20(10 - y), & 5 \leq y \leq 10. \end{cases}$$

Find the steady state temperature in the plate.

Ans:$u(x, y) = \frac{800}{\pi^2} \sum_{n=odd}^{\infty} \frac{1}{n^2} e^{\frac{n\pi x}{10}} \sin \frac{n\pi y}{10}$

## Two Marks

Note:

In 2nd order PDE,

1.If $B^2 - 4AC = 0$ then it is said to Prabolic

2.If $B^2 - 4AC < 0$ then it is said to Elliptic

3.If $B^2 - 4AC > 0$ then it is said to Hyperbolic.

1. Find the nature of the PDE

$x^2 u_{xx} + 2xyy_{xy} + (1 + y^2)u_{yy} - 2u_x = 0.$

Solution:

Given $x^2 u_{xx} + 2xyy_{xy} + (1 + y^2)u_{yy} - 2u_x = 0.$

Here $A = x^2, B = 2xy, C = 1 + y^2$

$\therefore B^2 - 4AC = 4x^2y^2 - 4x^2(1 + y^2)$

$= -4x^2$ If $x = 0$ then the equation is Prabolic

If $x < 0$ or $x > 0$ then the equation is Elliptic.

Classify the PDE,

(a) $y^2 u_{xx} - 2xyy_{xy} + x^2 u_{yy} + 2u_x - 3u_y = 0.$[Ans: Parabolic]

(b) $3\frac{\partial^2 u}{\partial x^2} + 43\frac{\partial^2 u}{\partial x \partial x} + 6\frac{\partial^2 u}{\partial y^2} - 2\frac{\partial u}{\partial y} - u = 0$.[Ans: Elliptic]

2. Write down the PDE governing the transverse vibrations of an elastic string.(or)
In the wave equation $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$,what is $c^2$ stands for?

Solution:

The PDE governing the transverse vibrations of an elastic string is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$$

where $c^2 = \frac{\text{Tension(T)}}{\text{Mass(M) per unit length of the string}}$.

3. State the governing equation for 1D heat equation.(or) In the diffusion equation $\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$,what is $\alpha^2$ stands for?

Solution:

The governing equation is

$$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$$

where $\alpha^2 = \frac{k}{\rho c}$

$k$-thermal conductivity

$\rho$-density of material

$c$-specific heat of the material

4. Write down the 2D heat flow equation in steady state.

Solution:

The 2D heat flow equation in steady state is the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

5. What are the various solutions of 1D wave equation:

Solution:

The possible solutions are

$$(i)y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat)$$
$$(ii)y(x,t) = (Ae^{px} + Be^{-px})(Ce^{pat} + De^{-pat})$$
$$(iii)y(x,t) = (Ax + B)(Ct + D)$$

6. State the three possible solutions of the heat equation $\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$

Solution:

The possible solutions are

$$(i)u(x,t) = (A\cos px + B\sin px)e^{-p^2\alpha^2 t}$$
$$(ii)u(x,t) = (Ae^{px} + Be^{-px})e^{p^2\alpha^2 t}$$
$$(iii)u(x,t) = Ax + B$$

7. Write down the possible solutions of the 2D heat equation in steady state. Solution:

   The possible solutions of the 2D heat equation in steady state will be:

$$(i)u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py})$$
$$(ii)u(x,y) = (Ae^{px} + Be^{-px})(C\cos py + D\sin py)$$
$$(iii)u(x,y) = (Ax + B)(Cy + D)$$

8. What is the basic difference between the solution of 1D wave and 1D heat equations?

   Solution:

   Solution of 1D wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{}{\partial x^2}$ is $y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat)$, which is **periodic** w.r.to t.

   But the Solution of 1D heat equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ is $u(x,t) = (A\cos px + B\sin px)e^{-p^2\alpha^2 t}$, which is **non-periodic** w.r.to t.

9. in steady state conditions derive the solution of 1D heat flow equation.

   Solution:

   The 1D heat equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$. .....(1)

   In steady state, $\frac{\partial u}{\partial t} = 0$

   then (1)$\Rightarrow \frac{\partial^2 u}{\partial x^2} = 0$

$$\Rightarrow \frac{\partial u}{\partial x} = a$$

$$\Rightarrow u = ax + b,$$ where a and b are arbitrary constants.

10. A rod 30 cm long has its ends A and B kept at $20 \circ C$ and $80 \circ C$ respectively until steady state condition prevail. Find the steady state temperature in the rod.

    Solution:

    The steady state temperature is

$$\begin{aligned} u(x) &= \frac{\theta_2 - \theta_1}{l}x + \theta_1, \text{Here } \theta_1 = 20, \theta_2 = 80, l = 30 \\ &= \frac{80-20}{30}x + 20 \\ \Rightarrow &= 2x + 20, \qquad 0 \le x \le 30 \end{aligned}$$

The ends A and B of a rod l cm long have temperature $40\circ C$ and $90\circ C$ until steady state prevails. Find the temperature in the rod that state.

Ans:$u(x) = \frac{50}{l}x + 40, \qquad 0 \le x \le l$

A rod 40cm long with insulated sides has its ends A and B kept at $20\circ C$ and $60\circ C$ respectively. Find the steady state temperature at a location 15cm for A.

Hint: $u(x) = \frac{x}{2} + 20, \qquad 0 \le x \le 40$

Sub $x = 15$, Ans:$27.5\circ C$

11. What conditions are assumed in deriving the 1D wave equation?

Solution:

Assumptions:

(i) The string is homogenous.

(ii) The string is perfectly elastic and so it does not offer any resistance to bending.

(iii) The tension T caused by stretching the string is so large so that the action of the gravitational force on the string can be neglected.

(iv) The string performs small transverse motions in a vertical plane so that the direction y and the slope $\frac{\partial y}{\partial x}$ are small in absolute value and hence, their higher powers may be neglected.

12. What are the laws assumed to derive the 1D heat equation?

Solution:

Assumptions:

(i) Heat flows from higher to lower temperature

(ii)The amount of heat required to produce a given temperature change in a body is proportional to the mass of the body and to the temperature change.

(iii) Fourier Law of Heat Conduction:

The rate at which heat flows through an area is proportional to the area and to the temperature gradient normal to the area.

Note:

In 2D they will ask the boundary conditions from the given question.

# Z Transform

Definition:1

Lef $\{f(n)\}$ be a sequence defined for $n = 0, \pm 1, \pm 2, \cdots$.Then the Z-transform is defined as

$$\begin{aligned} Z[f(n)] &= \sum_{n=-\infty}^{\infty} f(n) z^{-n}, && [z \rightarrow \text{a complex number}] \\ &= F(z) \end{aligned}$$

This is called **two sided or bilateral Z-transform.**

Definition:2

$$\begin{aligned} Z[f(n)] &= \sum_{n=0}^{\infty} f(n) z^{-n}, && [z \rightarrow \text{a complex number}] \\ &= F(z) \end{aligned}$$

This is called **one sided or unilateral Z-transform.**

1. Find

   (a) $Z[a^n]$

   Solution:

   Given $f(n) = a^n$

$$\begin{aligned} Z[f(n)] &= \sum_{n=0}^{\infty} f(n) z^{-n} \\ Z[a^n] &= \sum_{n=0}^{\infty} a^n z^{-n} \\ &= \sum_{n=0}^{\infty} (a z^{-1})^n \\ &= 1 + a z^{-1} + (a z^{-1})^2 + (a z^{-1})^3 + \cdots \\ &= (1 - a z^{-1})^{-1} && \text{if } |a z^{-1}| < 1 \\ &= \frac{1}{1 - \frac{a}{z}} && \text{if } |\frac{a}{z}| < 1 \\ &= \frac{z}{z - a} && \text{if } |z| > |a| \end{aligned}$$

   Note:

$$\text{If } a = 1, \quad Z[1] = \frac{z}{z-1} \quad \text{if } |z| > 1$$

$$\text{If } a = -1, \quad Z[-1] = \frac{z}{z+1} \quad \text{if } |z| > 1$$

   i. Find $Z[1]$

   ii. Find $Z[-1]$

iii. Find $\left(\frac{-1}{3}\right)^n$

(b) $Z[n]$

Solution:

Given $f(n) = n$

$$\begin{aligned}
Z[f(n)] &= \sum_{n=0}^{\infty} f(n) z^{-n} \\
Z[n] &= \sum_{n=0}^{\infty} n z^{-n} \\
&= 0 + 1 \cdot z^{-1} + 2 \cdot z^{-2} + 3 \cdot z^{-3} + \cdots \\
&= \frac{1}{z} + 2\frac{1}{z^2} + 3\frac{1}{z^3} + \cdots \\
&= \frac{1}{z}\left[1 + 2\frac{1}{z} + 3\frac{1}{z} + \cdots\right] \\
&= \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-2} \qquad \text{if } |z| > 1 \\
&= \frac{1}{z}\left(\frac{z-1}{z}\right)^{-2} \\
&= \frac{1}{z}\left(\frac{z^2}{(z-1)^2}\right) \\
&= \frac{z}{(z-1)^2} \qquad \text{if } |z| > 1
\end{aligned}$$

(c) $Z\left[\frac{1}{n}\right]$

Solution:

Given $f(n) = \frac{1}{n}$

$$\begin{aligned}
Z[f(n)] &= \sum_{n=0}^{\infty} f(n) z^{-n} \\
Z\left[\frac{1}{n}\right] &= \sum_{n=1}^{\infty} \frac{1}{n} z^{-n} \\
&= z^{-1} + \frac{z^{-2}}{2} + \frac{z^{-3}}{3} + \cdots \\
&= -\log\left(1 - z^{-1}\right) \qquad \text{if } |z| > 1 \\
&= -\log\left(1 - \frac{1}{z}\right) \\
&= -\log\left(\frac{z-1}{z}\right) \\
&= \log_e \frac{z}{z-1} \qquad \text{if } |z| > 1
\end{aligned}$$

(d) $Z\left[\frac{1}{n!}\right]$

Solution:

Given $f(n) = \frac{1}{n!}$

$$\begin{aligned} Z[f(n)] &= \sum_{n=0}^{\infty} f(n) z^{-n} \\ Z\left[\frac{1}{n}\right] &= \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} \\ &= 1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \cdots \\ &= e^{z^{-1}} \\ &= e^{\frac{1}{z}} \end{aligned}$$

(e) $Z\left[\frac{1}{(n+1)!}\right]$
Solution:

Given $f(n) = \frac{1}{(n+1)!}$

$$\begin{aligned} Z[f(n)] &= \sum_{n=0}^{\infty} f(n) z^{-n} \\ Z\left[\frac{1}{n}\right] &= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} z^{-n} \\ &= \frac{1}{1!} + \frac{z^{-1}}{2!} + \frac{z^{-2}}{3!} + \frac{z^{-3}}{4!} + \cdots \\ &= z\left[\frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \cdots\right] \\ &= z\left[e^{z^{-1}} - 1\right] \\ &= z\left[e^{\frac{1}{z}} - 1\right] \end{aligned}$$

(f) $Z\left[\frac{a^n}{n!}\right]$
Solution:

Given $f(n) = \frac{a^n}{n!}$

$$\begin{aligned} Z[f(n)] &= \sum_{n=0}^{\infty} f(n) z^{-n} \\ Z\left[\frac{a^n}{n}\right] &= \sum_{n=0}^{\infty} \frac{a^n}{n!} z^{-n} \\ &= \sum_{n=0}^{\infty} \frac{(az^{-1})^n}{n!} \\ &= 1 + \frac{az^{-1}}{1!} + \frac{(az^{-1})^2}{2!} + \frac{(az^{-1})^3}{3!} + \cdots \\ &= e^{az^{-1}} \\ &= e^{\frac{a}{z}} \end{aligned}$$

i. Find $Z[na^n]$

**Property:Differentiation in Z-Domain**

2. Prove that $Z[nf(n)] = -z\frac{d}{dz}Z[f(n)] = -z\frac{d}{dz}F(z)$.

Solution:

$$\begin{aligned} \text{W.K.T } F(z) &= Z[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n} \\ \frac{d}{dz}F(z) &= \frac{d}{dz}\sum_{n=0}^{\infty} f(n)z^{-n} \\ &= \sum_{n=0}^{\infty} f(n)(-n)z^{-n-1} \\ &= -\sum_{n=0}^{\infty} nf(n)z^{-n}\frac{1}{z} \\ -z\frac{d}{dz}F(z) &= \sum_{n=0}^{\infty} nf(n)z^{-n} \\ Z[nf(n)] &= -z\frac{d}{dz}F(z) \end{aligned}$$

(a) Find $Z[n^2]$

Solution:

$$\begin{aligned} Z[n^2] &= Z[n \times n] \\ &= -z\frac{d}{dz}Z[n] \\ &= -z\frac{d}{dz}\frac{z}{(z-1)^2} \\ &= -z\left[\frac{(z-1)^2(1) - z2(z-1)(1)}{(z-1)^4}\right] \\ &= -z\left[\frac{(z-1)(z-1-2z}{(z-1)^4}\right] \\ &= -z\left[\frac{-z-1}{(z-1)^3}\right] \\ &= \frac{z^2+z}{(z-1)^3} \end{aligned}$$

Show that $Z[n^3] = \frac{z^3+4z^2+z}{(z-1)^4}$

(b) Find $Z[na^n]$

Solution:

$$\begin{aligned} Z[na^n] &= -z\frac{d}{dz}Z[a^n] \\ &= -z\frac{d}{dz}\frac{z}{z-a} \\ &= -z\frac{(z-a)(1) - z(1)}{(z-a)^2} \\ &= \frac{az}{(z-a)^2} \end{aligned}$$

(c) Find $Z[(n+1)(n+2)]$

Solution:

$$\begin{aligned}
Z[(n+1)(n+2)] &= Z[n^2+3n+2] \\
&= Z[n^2]+3Z[n]+2Z[1] \\
&= \frac{z^2+z}{(z-1)^3}+3\frac{z}{(z-1)^2}+2\frac{z}{z-1} \\
&= \frac{z^2+z+3z(z-1)+2z(z-1)^2}{(z-1)^3} \\
&= \frac{z^2+z+3z^2-3z+2z(z^2-2z+1)}{(z-1)^3} \\
&= \frac{z^2+z+3z^2-3z+2z^3-4z^2+2z}{(z-1)^3} \\
&= \frac{2z^3}{(z-1)^3}
\end{aligned}$$

(d) Find $Z[n(n-1)(n-2)]$

Solution:

$$\begin{aligned}
Z[n(n-1)(n-2)] &= Z[n^3-3n^2+2n] \\
&= Z[n^3]-3Z[n^2]+2Z[n] \\
&= \frac{z^3+4z^2+z}{(z-1)^4}-3\frac{z^2+z}{(z-1)^3}+2\frac{z}{(z-1)^2} \\
&= \frac{z^3+4z^2+z-3(z^2+z)(z-1)+2z(z-1)^2}{(z-1)^4} \\
&= \frac{z^3+4z^2+z+(-3z^2-3z)(z-1)+2z(z^2-2z+1)}{(z-1)^4} \\
&= \frac{z^3+4z^2+z-3z^3-3z^2+3z^2+3z+2z^3-4z^2+2z}{(z-1)^4} \\
&= \frac{6z}{(z-1)^4}
\end{aligned}$$

3. Find $Z[a^n \cos n\theta]$ and $Z[a^n \sin n\theta]$

Solution:

$$\begin{aligned}
\text{W.K.T } e^{i\theta} &= \cos\theta + i\sin\theta \\
\text{and } e^{in\theta} &= \cos n\theta + i\sin n\theta \\
\text{Also } Z[a^n] &= \frac{z}{z-a} \\
\Rightarrow Z[(ae^{i\theta})^n] &= \frac{z}{z-ae^{i\theta}} \\
\Rightarrow Z[a^n\cos n\theta + ia^n \sin n\theta] &= \frac{z}{z-a(\cos\theta+i\sin\theta)} \\
\Rightarrow Z[a^n\cos n\theta] + iZ[a^n \sin n\theta] &= \frac{z}{(z-a\cos\theta)-ia\sin\theta} \\
&= \frac{z}{(z-a\cos\theta)-ia\sin\theta}\,\frac{(z-a\cos\theta)+ia\sin\theta}{(z-a\cos\theta)+ia\sin\theta}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{z[(z-a\cos\theta)+ia\sin\theta]}{(z-a\cos\theta)^2+a^2\sin^2\theta} \\
&= \frac{z(z-a\cos\theta)+iza\sin\theta}{z^2-2za\cos\theta+a^2\cos^2\theta+a^2\sin^2\theta} \\
&= \frac{z(z-a\cos\theta)+iza\sin\theta}{z^2-2za\cos\theta+a^2} \\
&= \frac{z(z-a\cos\theta)}{z^2-2za\cos\theta+a^2}+i\frac{za\sin\theta}{z^2-2za\cos\theta+a^2} ......(1) \\
\text{Similarly } Z[a^n\cos n\theta]-iZ[a^n\sin n\theta] &= \frac{z(z-a\cos\theta)}{z^2-2za\cos\theta+a^2}-i\frac{za\sin\theta}{z^2-2za\cos\theta+a^2} ......(2)
\end{aligned}
$$

$$
\begin{aligned}
(1)+(2) \quad &\Rightarrow \quad 2Z[a^n\cos n\theta] = \frac{2z(z-a\cos\theta)}{z^2-2za\cos\theta+a^2} \\
&\Rightarrow \quad Z[a^n\cos n\theta] = \frac{z(z-a\cos\theta)}{z^2-2za\cos\theta+a^2} \\
(1)-(2) \quad &\Rightarrow \quad 2Z[a^n\sin n\theta] = \frac{2za\sin\theta}{z^2-2za\cos\theta+a^2} \\
&\Rightarrow \quad Z[a^n\sin n\theta] = \frac{za\sin\theta}{z^2-2za\cos\theta+a^2}
\end{aligned}
$$

Note:

Find $Z[2^n\cos\frac{n\pi}{2}]$

$$
\begin{aligned}
\text{W.K.T } Z[a^n\cos n\theta] &= \frac{z(z-a\cos\theta)}{z^2-2za\cos\theta+a^2} \\
\text{put } a=2 \text{ and } \theta=\frac{\pi}{2} & \\
Z[2^n\cos n\frac{\pi}{2}] &= \frac{z(z-2\cos\frac{\pi}{2})}{z^2-2z(2)\cos\frac{\pi}{2}+4} \\
Z[2^n\cos\frac{n\pi}{2}] &= \frac{z^2}{z^2+4}
\end{aligned}
$$

Find $Z[2^n\sin\frac{n\pi}{2}]$

4. Find $Z[\cos n\theta]$ and $Z[\sin n\theta]$

Hint:Put $a=1$

$$
\begin{aligned}
Ans: Z[\cos n\theta] &= \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} \\
Z[\sin n\theta] &= \frac{z\sin\theta}{z^2-2z\cos\theta+1}
\end{aligned}
$$

Find $Z[\cos\frac{n\pi}{2}]$ and $Z[\sin\frac{n\pi}{2}]$

Note:1

$$
\begin{aligned}
1.\ Z[e^{-at}f(t)] &= Z[f(t)]_{z\to ze^{aT}} \\
2.\ Z[a^nf(t)] &= Z[f(t)]_{z\to\frac{z}{a}}
\end{aligned}
$$

Note:2

Find $Z[na^n]$

Solution:

$$\begin{aligned} Z[na^n] &= Z[n]_{z\to\frac{z}{a}} \\ &= \frac{z}{(z-1)^2}\Big|_{z\to\frac{z}{a}} \\ &= \frac{\frac{z}{a}}{\left(\frac{z}{a}-1\right)^2} \\ &= \frac{\frac{z}{a}}{\left(\frac{z-a}{a}\right)^2} \\ &= \frac{z}{a}\frac{a^2}{(z-a)^2} \\ &= \frac{az}{(z-a)^2} \end{aligned}$$

Try $Z[a^n\cos n\theta]$ and $Z[a^n\sin n\theta]$

Find $Z[e^{-at}\sin bt]$

$$\begin{aligned} \text{W.K.T } Z[e^{-at}f(t)] &= Z[f(t)]_{z\to ze^{aT}} \\ Z[e^{-at}\sin bt] &= Z[\sin bt]_{z\to ze^{aT}} \\ &= Z[\sin bnT]_{z\to ze^{aT}} \\ &= \frac{z\sin bT}{z^2-2z\cos bT+1}\Big|_{z\to ze^{aT}} \\ &= \frac{ze^{aT}\sin bT}{z^2e^{2aT}-2ze^{aT}\cos bT+1} \end{aligned}$$

5. Find the Z-transform of $\frac{1}{(n+1)(n+2)}$

Solution:

$$\begin{aligned} \text{Let } \frac{1}{(n+1)(n+2)} &= \frac{A}{n+1}+\frac{B}{n+2} \\ 1 &= A(n+2)+B(n+1) \end{aligned}$$

Put $z=-1$

$\Rightarrow 1 = A$

Put $z=-2$

$\Rightarrow 1=-B \Rightarrow B=-1$

$$\frac{1}{(n+1)(n+2)} = \frac{1}{n+1}-\frac{1}{n+2}$$

$$
\begin{aligned}
Z\left[\frac{1}{(n+1)(n+2)}\right] &= Z\left[\frac{1}{n+1}\right]-Z\left[\frac{1}{n+2}\right] \text{......}(1)\\
\text{Now } Z\left[\frac{1}{n+1}\right] &= \sum_{n=0}^{\infty}\frac{1}{n+1}z^{-n}\\
&= 1+\frac{z^{-1}}{2}+\frac{z^{-2}}{3}+\frac{z^{-3}}{4}+\cdots\\
&= z\left[\frac{z^{-1}}{1}+\frac{z^{-2}}{2}+\frac{z^{-3}}{3}+\frac{z^{-4}}{4}+\cdots\right]\\
&= -z\log\left(1-z^{-1}\right) \qquad \text{if } |z|>1\\
&= -z\log\left(1-\frac{1}{z}\right)\\
&= -z\log\left(\frac{z-1}{z}\right)\\
&= z\log_e\frac{z}{z-1} \qquad \text{if } |z|>1\\
\text{Also } Z\left[\frac{1}{n+2}\right] &= \sum_{n=0}^{\infty}\frac{1}{n+2}z^{-n}\\
&= \frac{1}{2}+\frac{z^{-1}}{3}+\frac{z^{-2}}{4}+\frac{z^{-3}}{5}+\cdots\\
&= z^2\left[\frac{z^{-2}}{2}+\frac{z^{-3}}{3}+\frac{z^{-4}}{4}+\cdots\right]\\
&= z^2\left[-\log\left(1-z^{-1}\right)-\frac{z^{-1}}{1}\right] \qquad \text{if } |z|>1\\
&= -z^2\log\left(1-\frac{1}{z}\right)+z\\
&= -z^2\log\left(\frac{z-1}{z}\right)+z\\
&= z^2\log_e\left(\frac{z}{z-1}\right)+z \qquad \text{if } |z|>1
\end{aligned}
$$

Sub in (1)

$$
\begin{aligned}
Z\left[\frac{1}{(n+1)(n+2)}\right] &= z\log_e\frac{z}{z-1}+z^2\log_e\left(\frac{z}{z-1}\right)+z\\
&= (z-z^2)\log_e\left(\frac{z}{z-1}\right)+z
\end{aligned}
$$

Find $Z\left[\dfrac{2n+3}{(n+1)(n+2)}\right]$

Ans: $\quad (z+z^2)\log_e\left(\dfrac{z}{z-1}\right)-z$

6. Prove that $Z[f(n+1)]=z[F(z)-f(0)]$[Second Shifting Theorem]

Solution:

$$
\begin{aligned}
F(z)=Z[f(n)] &= \sum_{n=0}^{\infty}f(n)z^{-n}\\
Z[f(n+1)] &= \sum_{n=0}^{\infty}f(n+1)z^{-n}\\
&= \sum_{n=0}^{\infty}f(n+1)z^{-n}z^{-1}z
\end{aligned}
$$

$$= z\sum_{n=0}^{\infty} f(n+1)z^{-(n+1)}$$

Put $n+1=m$,

if $n=0 \Rightarrow m=1$

if $n=\infty \Rightarrow m=\infty$

$$\begin{aligned}
\text{we get,} \qquad Z[f(n+1)] &= z\sum_{m=1}^{\infty} f(m)z^{m} \\
&= z\sum_{m=1}^{\infty} f(m)z^{m} + zf(0)z^{-0} - zf(0)z^{-0} \\
&= z\left[\sum_{m=1}^{\infty} f(m)z^{m} + f(0)z^{-0}\right] - zf(0) \\
&= z\sum_{m=0}^{\infty} f(m)z^{m} - zf(0) \\
&= zF(z) - zf(0)
\end{aligned}$$

Note:

If $Z[f(n)] = f(z)$ then find $Z[f(n-k)]$ and $Z[f(n+k)]$

**Initial and Final value Theorem:**

7. State and prove the Initial and final value theorem.

Solution:

Initial value theorem:

If $Z[f(n)] = f(z)$ then $f(0) = \lim_{z\to\infty} F(z)$

$$\begin{aligned}
\text{W.K.T} \qquad F(z) &= Z[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n} \\
&= f(0) + f(1)z^{-1} + f(2)z^{-2} + f(3)z^{-3} + \cdots \\
&= f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \frac{f(3)}{z^3} + \cdots \\
\lim_{z\to\infty} F(z) &= \lim_{z\to\infty}\left[f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \frac{f(3)}{z^3} + \cdots\right] \\
&= f(0)
\end{aligned}$$

Final value theorem:

If $Z[f(n)] = f(z)$ then $\lim_{n\to\infty} f(n) = \lim_{z\to 1}(z-1)F(z)$

$$\begin{aligned}
\text{W.K.T} \qquad Z[f(n+1)] &= z[F(z) - f(0)] \\
&= zF(z) - zf(0) \\
Z[f(n+1)] - F(z) &= zF(z) - zf(0) - F(z) \\
Z[f(n+1)] - Z[f(n)] &= (z-1)F(z) - zf(0)
\end{aligned}$$

$$\begin{aligned}
Z[f(n+1)-f(n)] &= (z-1)F(z)-zf(0)\\
\sum_{n=0}^{\infty}[f(n+1)-f(n)]z_{-n} &= (z-1)F(z)-zf(0)\\
\lim_{z\to 1}(z-1)F(z)-f(0) &= \lim_{n\to\infty}[f(1)-f(0)+f(2)-f(1)+f(3)-f(2)+\cdots+f(n+1)-f(n)]\\
&= \lim_{n\to\infty}[f(n)-f(0)]\\
\lim_{z\to 1}(z-1)F(z)-f(0) &= \lim_{n\to\infty}[f(n)-f(0)]\\
\lim_{z\to 1}(z-1)F(z) &= \lim_{n\to\infty} f(n)
\end{aligned}$$

## Convolution Theorem

Note:

1. $1+a+a^2+a^3+\ldots+a^n = \dfrac{a^{n+1}-1}{a-1}$ if$a>1$
2. $1+a+a^2+a^3+\ldots+a^n = \dfrac{1-a^{n+1}}{1-a}$ if$a<1$

8. Using Convolution theorem, find the $Z^{-1}\left[\dfrac{z^2}{(z-a)(z-b)}\right]$

Solution:

$$\begin{aligned}
Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] &= Z^{-1}\left[\frac{z}{z-a}\right]Z^{-1}\left[\frac{z}{z-b}\right]\\
&= a^n * b^n\\
&= \sum_{r=0}^{n} a^r b^{n-r}\\
&= b^n\sum_{r=0}^{n}\left(\frac{a}{b}\right)^r\\
&= b^n\left[1+\frac{a}{b}+\left(\frac{a}{b}\right)^2+\left(\frac{a}{b}\right)^3+\ldots+\left(\frac{a}{b}\right)^n\right]\\
&= b^n\left[\frac{\left(\frac{a}{b}\right)^{n+1}-1}{\frac{a}{b}-1}\right]\\
&= b^n\left[\frac{\frac{a^{n+1}-b^{n+1}}{b^{n+1}}}{\frac{a-b}{b}}\right]\\
&= b^n\left(\frac{a^{n+1}-b^{n+1}}{b^{n+1}}\right)\frac{b}{a-b}\\
&= \frac{a^{n+1}-b^{n+1}}{a-b}
\end{aligned}$$

Using Convolution theorem, find the $Z^{-1}\left[\dfrac{z^2}{(z-1)(z-3)}\right]$

9. Using Convolution theorem, find the $Z^{-1}\left[\dfrac{8z^2}{(2z-1)(4z+1)}\right]$

Solution:

$$\begin{aligned}
Z^{-1}\left[\frac{8z^2}{2\left(z-\frac{1}{2}\right)4\left(z+\frac{1}{4}\right)}\right] &= Z^{-1}\left[\frac{z}{\left(z-\frac{1}{2}\right)}\right] Z^{-1}\left[\frac{z}{\left(z-\frac{-1}{4}\right)}\right] \\
&= \left(\frac{1}{2}\right)^n * \left(\frac{-1}{4}\right)^n \\
&= \sum_{r=0}^{n}\left(\frac{-1}{4}\right)^r\left(\frac{1}{2}\right)^{n-r} \\
&= \left(\frac{1}{2}\right)^n\sum_{r=0}^{n}\left(\frac{-1}{4}\right)^r\left(\frac{1}{2}\right)^{-r} \\
&= \left(\frac{1}{2}\right)^n\sum_{r=0}^{n}\left(\frac{-1}{4}\right)^r 2^r \\
&= \left(\frac{1}{2}\right)^n\sum_{r=0}^{n}\left(\frac{-1}{4}\dot{2}\right)^r \\
&= \left(\frac{1}{2}\right)^n\sum_{r=0}^{n}\left(\frac{-1}{2}\right)^r \\
&= \left(\frac{1}{2}\right)^n\left[1+\frac{-1}{2}+\left(\frac{-1}{2}\right)^2+\left(\frac{-1}{2}\right)^3+\ldots+\left(\frac{-1}{2}\right)^n\right] \\
&= \left(\frac{1}{2}\right)^n\left[\frac{1-\left(\frac{-1}{2}\right)^{n+1}}{1-\left(\frac{-1}{2}\right)}\right] \\
&= \left(\frac{1}{2}\right)^n\left[\frac{1-\left(\frac{-1}{2}\right)^{n+1}}{\frac{3}{2}}\right] \\
&= \frac{2}{3}\left(\frac{1}{2}\right)^n\left[1-\left(\frac{-1}{2}\right)\left(\frac{-1}{2}\right)^n\right] \\
&= \frac{2}{3}\left(\frac{1}{2}\right)^n\left[1+\frac{1}{2}\left(\frac{-1}{2}\right)^n\right] \\
&= \frac{2}{3}\left(\frac{1}{2}\right)^n+\frac{1}{3}\left(\frac{-1}{4}\right)^n
\end{aligned}$$

Using Convolution theorem, find the $Z^{-1}\left[\frac{8z^2}{(2z-1)(4z-1)}\right]$

Using Convolution theorem, find the $Z^{-1}\left[\frac{14z^2}{(7z-1)(2z-1)}\right]$

10. Find $Z^{-1}\left[\frac{z^2}{(z-a)^2}\right]$ using convolution theorem.

Solution:

$$\begin{aligned}
Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] &= Z^{-1}\left[\frac{z}{z-a}\right] Z^{-1}\left[\frac{z}{z-a}\right] \\
&= a^n * a^n
\end{aligned}$$

$$\begin{aligned}
&= \sum_{r=0}^{n} a^r a^{n-r} \\
&= \sum_{r=0}^{n} a^n \\
&= a^n \sum_{r=0}^{n} 1 \\
&= a^n[1+1+1+\cdots+1] \\
&= (n+1)a^n
\end{aligned}$$

11. Find $Z^{-1}\left[\dfrac{z^3}{(z-4)^3}\right]$ using convolution theorem.

Solution:

$$\begin{aligned}
Z^{-1}\left[\frac{z^3}{(z-4)^3}\right] &= Z^{-1}\left[\frac{z^2}{(z-4)^2}\right] Z^{-1}\left[\frac{z}{z-4}\right] \\
&= (n+1)4^n * 4^n \\
&= \sum_{r=0}^{n}(r+1)4^r 4^{n-r} \\
&= \sum_{r=0}^{n}(r+1)4^n \\
&= 4^n \sum_{r=0}^{n}(r+1) \\
&= a^n[1+2+3+\cdots+(n+1)] \\
&= \frac{(n+1)(n+2)}{2}4^n
\end{aligned}$$

12. Find $Z^{-1}\left[\dfrac{z^3}{(z-2)^2(z-3)}\right]$

Solution:

$$\begin{aligned}
Z^{-1}\left[\frac{z^3}{(z-2)^2(z-3)}\right] &= Z^{-1}\left[\frac{z^2}{(z-2)^2}\right] Z^{-1}\left[\frac{z}{z-3}\right] \\
&= (n+1)2^n * 3^n \\
&= \sum_{r=0}^{n}(r+1)2^r 3^{n-r} \\
&= 3^n \sum_{r=0}^{n}(r+1)\left(\frac{2}{3}\right)^r \\
&= 3^n\left[1+2\left(\frac{2}{3}\right)+3\left(\frac{2}{3}\right)^2+4\left(\frac{2}{3}\right)^3+\ldots+(n+1)\left(\frac{2}{3}\right)^n\right] \\
\text{Let } S &= 1+2x+3x^2+4x^3+\ldots+nx^{n-1}+(n+1)x^n, \text{where } x=\frac{2}{3} \\
xS &= x+2x^2+3x^3+4x^4+\ldots+nx^n+(n+1)x^{n+1} \\
(1-x)S &= (1+x+x^2+x^3+\ldots+x^n)-(n+1)x^{n+1} \\
&= \frac{1-x^{n+1}}{1-x}-(n+1)x^{n+1} \\
\therefore S &= \frac{1-x^{n+1}}{(1-x)^2}-\frac{(n+1)x^{n+1}}{1-x}
\end{aligned}$$

Put $x = \frac{2}{3} \Rightarrow 1 - x = \frac{1}{3}$

$$\begin{aligned}
\therefore S &= \frac{1 - \left(\frac{2}{3}\right)^{n+1}}{\left(\frac{2}{3}\right)^2} - \frac{(n+1)\left(\frac{2}{3}\right)^{n+1}}{\left(\frac{2}{3}\right)} \\
&= 9\left[1 - \left(\frac{2}{3}\right)^{n+1}\right] - 3(n+1)\left(\frac{2}{3}\right)^{n+1} \\
&= 9 - \left(\frac{2}{3}\right)^{n+1} - 3(n+1)\left(\frac{2}{3}\right)^{n+1} \\
&= 9 - \left(\frac{2}{3}\right)^{n+1}[9 + 3n + 3] \\
&= 9 - \left(\frac{2}{3}\right)^{n+1}[12 + 3n] \\
&= 9 - 3(n+4)\left(\frac{2}{3}\right)^{n+1}
\end{aligned}$$

$$\begin{aligned}
Z^{-1}\left[\frac{z^3}{(z-2)^2(z-3)}\right] &= 3^n\left[9 - 3(n+4)\left(\frac{2}{3}\right)^{n+1}\right] \\
&= 9 \cdot 3^n - 3^{n+1}(n+4)\left(\frac{2}{3}\right)^{n+1} \\
&= 3^{n+2} - (n+4)2^{n+1}
\end{aligned}$$

## Method of Partial fraction

Note:

$$Type1: \quad \frac{f(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

$$Type2: \quad \frac{f(x)}{(x-a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$$

$$Type3: \quad \frac{f(x)}{(x-a)(x^2-bx-c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2-bx-c}$$

13. Find $Z^{-1}\left[\frac{10z}{z^2-3z+2}\right]$

Solution:

$$\begin{aligned}
\text{Let } F(z) &= \frac{10z}{z^2-3z+2} \\
\Rightarrow \frac{F(z)}{z} &= \frac{10}{(z-1)(z-2)} \\
\text{Let } \frac{10}{(z-1)(z-2)} &= \frac{A}{z-1} + \frac{B}{z-2} \\
10 &= A(z-2) + B(z-1)
\end{aligned}$$

Put $z = 1$

$\Rightarrow 10 = -A$

$A = -10$

Put $z = 2$

$\Rightarrow 10 = B$

$$\begin{aligned}
\frac{10}{(z-1)(z-2)} &= \frac{-10}{z-1} + \frac{10}{z-2} \\
\frac{F(z)}{z} &= \frac{-10}{z-1} + \frac{10}{z-2} \\
F(z) &= -10\frac{z}{z-1} + 10\frac{z}{z-2} \\
f(n) &= -10Z^{-1}\left[\frac{z}{z-1}\right] + 10Z^{-1}\left[\frac{z}{z-2}\right] \\
&= -10(1)^n + 10 \cdot 2^n
\end{aligned}$$

Evaluate $Z^{-1}\left[\frac{z}{z^2+7z+10}\right]$

Find the inverse Z-transform of $\frac{2z^2+3z}{(z+2)(z-4)}$

14. Find $Z^{-1}\left[\frac{z^3}{(z-1)^2(z-2)}\right]$

Solution:

$$\begin{aligned}
\text{Let } F(z) &= \frac{z^3}{(z-1)^2(z-2)} \\
\Rightarrow \frac{F(z)}{z} &= \frac{z^2}{(z-1)^2(z-2)} \\
\text{Let } \frac{z^2}{(z-1)^2(z-2)} &= \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z-2} \\
z^2 &= A(z-1)(z-2) + B(z-2) + C(z-1)^2
\end{aligned}$$

Put $z = 1$

$\Rightarrow 1 = -B$

$B = -1$

Put $z = 2$

$\Rightarrow 4 = C$

Eq.coeff of $z^2$

$1 = A + C \Rightarrow A = 1 - C$

$\Rightarrow A = 1 - 4$

$\Rightarrow A = -3$

$$\begin{aligned}
\frac{z^2}{(z-1)^2(z-2)} &= \frac{-3}{z-1} + \frac{-1}{(z-1)^2} + \frac{4}{z-2} \\
\frac{F(z)}{z} &= \frac{-3}{z-1} + \frac{-1}{(z-1)^2} + \frac{4}{z-2} \\
F(z) &= -3\frac{z}{z-1} - \frac{z}{(z-1)^2} + 4\frac{z}{z-2}
\end{aligned}$$

$$
\begin{aligned}
f(n) &= -3Z^{-1}\left[\frac{z}{z-1}\right] - Z^{-1}\left[\frac{z}{(z-1)^2}\right] + 4Z^{-1}\left[\frac{z}{z-2}\right] \\
&= -3(1)^n - n + 4\cdot 2^n
\end{aligned}
$$

Find $Z^{-1}\left[\frac{z}{(z-1)^2(z+1)}\right]$

Find $Z^{-1}\left[\frac{z(z^2-z+2)}{(z-1)^2(z+1)}\right]$

15. Find $Z^{-1}\left[\frac{z^2}{(z+2)(z^2+4)}\right]$ by the method of partial fraction.

Solution:

$$
\begin{aligned}
\text{Let } F(z) &= \frac{z^2}{(z+2)(z^2+4)} \\
\Rightarrow \frac{F(z)}{z} &= \frac{z}{(z+2)(z^2+4)} \\
\text{Let } \frac{z}{(z+2)(z^2+4)} &= \frac{A}{z+2} + \frac{Bz+C}{z^2+4} \\
z &= A(z^2+4) + (Bz+C)(z+2)
\end{aligned}
$$

Put $z = -2$

$\Rightarrow -2 = 8A$

$A = -\frac{1}{4}$

Put $z = 0$

$\Rightarrow 0 = 4A + 2C$

$\Rightarrow 2C = -4\frac{-1}{4}$

$\Rightarrow C = \frac{1}{2}$

Eq.coeff of $z^2$

$0 = A + B \Rightarrow B = -A$

$B = \frac{1}{4}$

$$
\begin{aligned}
\frac{z}{(z+2)(z^2+4)} &= \frac{\frac{-1}{4}}{z+2} + \frac{\frac{1}{4}z + \frac{1}{2}}{z^2+4} \\
\frac{F(z)}{z} &= -\frac{1}{4}\frac{1}{z+2} + \frac{1}{4}\frac{z}{z^2+4} + \frac{1}{2}\frac{1}{z^2+4} \\
F(z) &= -\frac{1}{4}\frac{z}{z-(-2)} + \frac{1}{4}\frac{z^2}{z^2+4} + \frac{1}{2}\frac{z}{z^2+4} \\
f(n) &= -\frac{1}{4}Z^{-1}\left[\frac{z}{z-(-2)}\right] + \frac{1}{4}Z^{-1}\left[\frac{z^2}{z^2+2^2}\right] + \frac{1}{4}Z^{-1}\left[\frac{2z}{z^2+2^2}\right] \\
&= -\frac{1}{4}(-2)^n + \frac{1}{4}2^n\cos\frac{n\pi}{2} + \frac{1}{4}2^n\sin\frac{n\pi}{2}
\end{aligned}
$$

Find $Z^{-1}\left[\frac{z^3+3z}{(z-1)^2(z^2+1)}\right]$

## Residue Method

Note:

Simple Pole:

$$\{ResF(z)z^{n-1}\}_{z=a} = \lim_{z\to a}(z-a)F(z)z^{n-1}$$

Pole of order n:

$$\{ResF(z)z^{n-1}\}_{z=a} = \frac{1}{(n-1)!}\lim_{z\to a}\frac{\partial^{n-1}}{\partial z^{n-1}}(z-a)^n F(z)z^{n-1}$$

16. Find $Z^{-1}\left[\frac{z^2-3z}{(z+2)(z-5)}\right]$ using residue method.

Solution

$$F(z) = \frac{z^2-3z}{(z+2)(z-5)}$$

$$z^{n-1}F(z) = z^{n-1}\frac{z(z-3)}{(z+2)(z-5)}$$

$$= \frac{z^n(z-3)}{(z+2)(z-5)}$$

Eq the dominator to zero

$$(z+2)(z-5) = 0$$

$$z = 5, -2$$

The poles are simple, $z = 5, -2$

W.K.T $\{ResF(z)z^{n-1}\}_{z=a} = \lim_{z\to a}(z-a)F(z)z^{n-1}$

When $z = 5$

$$\{ResF(z)z^{n-1}\}_{z=5} = \lim_{z\to 5}(z-5)\frac{z^n(z-3)}{(z+2)(z-5)}$$

$$= \frac{5^n(2)}{7}$$

$$= \frac{2}{7}5^n$$

When $z = -2$

$$\{ResF(z)z^{n-1}\}_{z=-2} = \lim_{z\to 5}(z+2)\frac{z^n(z-3)}{(z+2)(z-5)}$$

$$= \frac{(-2)^n(-5)}{-7}$$

$$= \frac{5}{7}(-2)^n$$

$$\therefore f(n) = \text{Sum of the Residues}$$

$$= \frac{2}{7}5^n + \frac{5}{7}(-2)^n$$

17. If $U(z) = \frac{2z^2+3z+12}{(z-1)^4}$, find the values of $u_2$ and $u_3$

Solution

$$U(z) = \frac{2z^2+3z+12}{(z-1)^4}$$

$$z^{n-1}U(z) = z^{n-1}\frac{2z^2+3z+12}{(z-1)^4}$$

Eq the dominінator to zero

$$(z-1)^4 = 0$$

$$z = 1,1,1,1$$

The poles are order 4, $z=1$

$$\text{W.K.T } \{ResF(z)z^{n-1}\}_{z=a} = \frac{1}{(n-1)!}\lim_{z\to a}\frac{\partial^{n-1}}{\partial z^{n-1}}(z-a)^n F(z)z^{n-1}$$

When $z=1$

$$\begin{aligned}
\{ResF(z)z^{n-1}\}_{z=1} &= \frac{1}{3!}\lim_{z\to 1}\frac{\partial^3 z}{\partial z^3}(z-1)^4 z^{n-1}\frac{2z^2+3z+12}{(z-1)^4} \\
&= \frac{1}{6}\lim_{z\to 1}\frac{\partial^3 z}{\partial z^3}[2z^{n+1}+3z^n+12z^{n-1}] \\
&= \frac{1}{6}\lim_{z\to 1}\frac{\partial^2 z}{\partial z^2}[2(n+1)z^n+3nz^{n-1}+12(n-1)z^{n-2}] \\
&= \frac{1}{6}\lim_{z\to 1}\frac{\partial z}{\partial z}[2(n+1)nz^{n-1}+3n(n-1)z^{n-2}+12(n-1)(n-2)z^{n-3}] \\
&= \frac{1}{6}\lim_{z\to 1}[2(n+1)n(n-1)z^{n-2}+3n(n-1)(n-2)z^{n-3}+12(n-1)(n-2)(n-3)z^{n-4}] \\
&= \frac{1}{6}[2(n+1)n(n-1)+3n(n-1)(n-2)+12(n-1)(n-2)(n-3)]
\end{aligned}$$

$$\begin{aligned}
\therefore f(n) &= \text{Sum of the Residues} \\
&= \frac{1}{6}[2(n+1)n(n-1)+3n(n-1)(n-2)+12(n-1)(n-2)(n-3)]
\end{aligned}$$

Put n=2, $u_2 = \frac{1}{6}[2\cdot 3\cdot 2\cdot 1] = 2$

Put n=3, $u_3 = \frac{1}{6}[2\cdot 4\cdot 3\cdot 2+3\cdot 3\cdot 2\cdot 1] = 11$

If $U(z) = \frac{2z^2+5z+14}{(z-1)^4}$, find the values of $u_2$ and $u_3$

Find $Z^{-1}\left[\frac{z(z+1)}{(z-1)^3}\right]$

## Formation of Differential Equation

18. Form a differential equation by eliminating arbitrary constants $y_n = a + b3^n$

Solution:

$$\begin{aligned}
y_n &= a + b3^n \\
y_{n+1} &= a + b3^{n+1} \\
&= a + 3b3^n \\
y_{n+2} &= a + b3^{n+2} \\
&= a + 9b3^n
\end{aligned}$$

Eliminating $a$ and $b3^n$,

$$\begin{aligned}
\begin{vmatrix} y_n & y_{n+1} & y_{n+2} \\ 1 & 1 & 1 \\ 1 & 3 & 9 \end{vmatrix} &= 0 \\
y_n[9-3] - y_{n+1}[9-1] + y_{n+2}[3-1] &= 0 \\
6y_n - 8y_{n+1} + 2y_{n+2} &= 0 \\
y_{n+2} - 4y_{n+1} + 3y_n &= 0
\end{aligned}$$

19. Form a differential equation by eliminating arbitrary constants $y_n = (A + Bn)2^n$

Solution:

$$\begin{aligned}
y_n &= A2^n + Bn2^n \\
y_{n+1} &= A2^{n+1} + B(n+1)2^{n+1} \\
&= 2A2^n + 2B(n+1)2^n \\
y_{n+2} &= A2^{n+2} + B(n+2)2^{n+2} \\
&= 4A2^n + 4B(n+2)2^n
\end{aligned}$$

Eliminating $a$ and $b3^n$,

$$\begin{aligned}
\begin{vmatrix} y_n & y_{n+1} & y_{n+2} \\ 1 & 2 & 4 \\ n & 2(n+1) & 4(n+2) \end{vmatrix} &= 0 \\
y_n[8(n+2) - 8(n+1)] - y_{n+1}[4(n+2) - 4n] + y_{n+2}[2(n+1) - 2n] &= 0 \\
8y_n - 8y_{n+1} + 2y_{n+2} &= 0 \\
y_{n+2} - 4y_{n+1} + 4y_n &= 0
\end{aligned}$$

Form a differential equation by eliminating arbitrary constants $y_n = a - b3^n$

Form a differential equation by eliminating arbitrary constants $y_n = an + b2^n$

## Solving Linear Differential equation

Formula:

$$\begin{aligned} Z[y_n] &= F(z) \\ Z[y_{n+1}] &= zF(z) - zy(0) \\ Z[y_{n+2}] &= z^2F(z) - z^2y(0) - zy(1) \\ Z[y_{n+3}] &= z^3F(z) - z^3y(0) - z^2y(1) - zy(2) \end{aligned}$$

20. Using Z-transform solve $u_{n+2} - 5u_{n+1} + 6u_n = 4^n$ given that $u_0 = 0, u_1 = 1$.

Solution:

Given $u_{n+2} - 5u_{n+1} + 6u_n = 4^n$

Applying Z-transforms on both side,

$$\begin{aligned} Z[u_{n+2}] - 5Z[u_{n+1}] + 6Z[u_n] &= Z[4^n] \\ z^2F(z) - z^2u(0) - zu(1) - 5[zF(z) - zu(0)] + 6F(z) &= \frac{z}{z-4} \\ \text{Given } u_0 = 0, u_1 = 1 \ F(z)[z^2 - 5z + 6] - z &= \frac{z}{z-4} \\ F(z)(z-2)(z-3) &= \frac{z}{z-4} + z \\ &= \frac{z + z^2 - 4z}{z-4} \\ F(z) &= \frac{z^2 - 3z}{(z-4)(z-2)(z-3)} \\ &= \frac{z(z-3)}{(z-4)(z-2)(z-3)} \\ &= \frac{z}{(z-4)(z-2)} \\ \text{By Residue Method,} & \\ z^{n-1}F(z) &= z^{n-1}\frac{z}{(z-4)(z-2)} \\ &= \frac{z^n(z-3)}{(z-4)(z-2)} \\ \text{Eq the domininator to zero} & \\ (z-4)(z-2) &= 0 \\ z &= 2, 4 \\ \text{The poles are simple,} z = 2, 4 & \end{aligned}$$

$$\text{W.K.T } \{ResF(z)z^{n-1}\}_{z=a} = \lim_{z\to a}(z-a)F(z)z^{n-1}$$

When $z=2$

$$\begin{aligned}\{ResF(z)z^{n-1}\}_{z=2} &= \lim_{z\to 2}(z-2)\frac{z^n(z-2)}{(z-4)(z-2)} \\ &= \frac{2^n}{-2} \\ &= -(2^{n-1})\end{aligned}$$

When $z=4$

$$\begin{aligned}\{ResF(z)z^{n-1}\}_{z=4} &= \lim_{z\to 4}(z-4)\frac{z^n(z-4)}{(z-4)(z-2)} \\ &= \frac{4^n}{2} \\ &= 2^{2n-1}\end{aligned}$$

$$\begin{aligned}\therefore u_n &= \text{Sum of the Residues} \\ &= -(2^{n-1}) + 2^{2n-1}\end{aligned}$$

Solve the differential equation $y(n+3)-3y(n+1)+2y(n)=0$ given that $y(0)=4, y(1)=0$ and $y(2)=8$.
Ans:$y(n)=\frac{8}{3}n+\frac{4}{3}\cdot(-2)^n$

Solve the differential equation $u_{n+2}+3u_{n+1}+2u_n=0$ given that $u_0=1, u_1=2$

Ans:$u_n=4(-1)^n-3(-2)^n$

Solve the differential equation $y_{n+2}+4y_{n+1}+3y_n=2^n$ given that $y_0=0, y_1=1$
Ans:$y_n=-\frac{2}{5}(-3)^n+\frac{1}{3}\cdot(-1)^n+\frac{1}{15}\cdot(2)^n$

Solve the differential equation $y(n)+3y(n-1)-4y(n-2)=0, n\geq 2$ given that $y(0)=3, y(1)=-2$

Hint:Changing $n$ to $n+2$

Ans:$y(n)=(-4)^n+2$

Solve the differential equation $y(n+3)-3y(n+1)+2y(n)=0$ with $y(0)=4, y(1)=0$ and $y(2)=8$
Ans:$y(n)=\frac{8}{3}+\frac{4}{3}(-2)^n$

21. Solve $y_{n+2}+6y_{n+1}+9y_n=2^n$ given that $y_0=y_1=0$ using Z-transform

    Solution:

Given $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$

Applying Z-transforms on both side,

$$
\begin{aligned}
Z[y_{n+2}] + 6Z[y_{n+1}] + 9Z[y_n] &= Z[2^n] \\
z^2F(z) - z^2y(0) - zy(1) + 6[zF(z) - zy(0)] + 9F(z) &= \frac{z}{z-2} \\
\text{Given } y_0 = y_1 = 0 \; F(z)[z^2 + 6z + 9] &= \frac{z}{z-2} \\
F(z)(z+3)^2 &= \frac{z}{z-2} \\
F(z) &= \frac{z}{(z-2)(z+3)^2} \\
\text{By Residue Method,} & \\
z^{n-1}F(z) &= z^{n-1}\frac{z}{(z-2)(z+3)^2} \\
&= \frac{z^n}{(z-2)(z+3)^2} \\
\text{Eq the domininator to zero} & \\
(z-2)(z+3)^2 &= 0 \\
z &= 2, -3, -3 \\
\text{The poles are simple,} z = 2 & \\
\text{The poles are order } 2, z = -3, 4 & \\
\text{W.K.T } \{ResF(z)z^{n-1}\}_{z=a} &= \lim_{z \to a}(z-a)F(z)z^{n-1}
\end{aligned}
$$

When $z = 2$

$$
\begin{aligned}
\{ResF(z)z^{n-1}\}_{z=2} &= \lim_{z \to 2}(z-2)\frac{z^n(z-2)}{(z-2)(z+3)^2} \\
&= \frac{2^n}{25}
\end{aligned}
$$

$\{ResF(z)z^{n-1}\}_{z=a} = \frac{1}{(n-1)!}\lim_{z \to a}\frac{\partial^{n-1}}{\partial z^{n-1}}(z-a)^n F(z)z^{n-1}$ When $z = -3$

$$
\begin{aligned}
\{ResF(z)z^{n-1}\}_{z=-3} &= \frac{1}{(1)!}\lim_{z \to -3}\frac{\partial z}{\partial z}(z+3)^2\frac{z^n}{(z-2)(z+3)^2} \\
&= \lim_{z \to -3}\frac{\partial z}{\partial z}[\frac{z^n}{z-2}] \\
&= \lim_{z \to -3}[\frac{(z-2)nz^{n-1} - z^n(1)}{(z-2)^2}] \\
&= \frac{-5n(-3)^{n-1} - (-3)^n}{25} \\
&= \frac{-5n\frac{(-3)^n}{-3} - (-3)^n}{25} \\
&= \frac{1}{15}n(-3)^n - \frac{1}{25}(-3)^n
\end{aligned}
$$

$$\begin{aligned}
\therefore u_n &= \text{Sum of the Residues} \\
&= \frac{2^n}{25} + \frac{1}{15} n(-3)^n - \frac{1}{25}(-3)^n
\end{aligned}$$

Solve by Z-transform $u_{n+2} - 2u_{n+1} + u_n = 2^n$ with $u_0 = 2$ and $u_1 = 1$.

Ans:$u_n = 2^n + 1 - 2n$

Solve the equation $y_{n+2} - 3y_{n+1} + 2y_n = 2^n$ with $y_0 = y_1 = 0$

Ans:$y_n = 1 + n2^{n-1} - 2^n$

Solve the differential equation $y(k+2) - 4y(k+1) + 4y(k) = 0$ with $y(0) = 1, y(1) = 0$

Ans:$y(n) = 2^k - k \cdot 2^k$

Solve the equation $y_{n+2} + 4y_{n+1} - 5y_n = 24n - 8$ with $y_0 = 3, y_1 = -5$

Ans:$y_n = (-5)^n + 2n^2 - 4n + 2$[use partial fraction method]

22. Solve the differential equation $y(k+2) + y(k) = 1$ with $y(0) = y(1) = 0$

Solution:

Given $y(n+2) + y(n) = 0$

Applying Z-transforms on both side,

$$\begin{aligned}
Z[y_{n+2}] + Z[y_n] &= Z[1] \\
z^2 F(z) - z^2 y(0) - zy(1) + F(z) &= \frac{z}{z-1} \\
\text{Given } y_0 = y_1 = 0 \;\; F(z)[z^2+1] &= \frac{z}{z-1} \\
F(z) &= \frac{z}{(z-1)(z^2+1)}
\end{aligned}$$

By Partial fraction Method,

$$\begin{aligned}
\Rightarrow \frac{F(z)}{z} &= \frac{1}{(z-1)(z^2+1)} \\
\text{Let } \frac{1}{(z-1)(z^2+1)} &= \frac{A}{z-1} + \frac{Bz+C}{z^2+1} \\
1 &= A(z^2+1) + (Bz+C)(z-1)
\end{aligned}$$

Put $z = 1$

$\Rightarrow 1 = 2A$

$A = \frac{1}{2}$

Put $z = 0$

$1 = A - C \Rightarrow C = A - 1$

$\Rightarrow C = \frac{1}{2} - 1$

$\Rightarrow C = -\frac{1}{2}$

Eq.coeff of $z^2$

$0 = A + B \Rightarrow B = -A$

$B = -\frac{1}{2}$

$$
\begin{aligned}
\therefore \frac{1}{(z-1)(z^2+1)} &= \frac{\frac{1}{2}}{z-1} + \frac{\frac{-1}{2}z + \frac{-1}{2}}{z^2+1} \\
\frac{F(z)}{z} &= \frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{z}{z^2+1} + \frac{1}{2}\frac{1}{z^2+1} \\
F(z) &= \frac{1}{2}Z^{-1}\left[\frac{z}{z-1}\right] - \frac{1}{2}Z^{-1}\left[\frac{z^2}{z^2+1}\right] - \frac{1}{2}Z^{-1}\left[\frac{z}{z^2+1}\right] \\
y(n) &= \frac{1}{2}\frac{z}{z-1} - \frac{1}{2}\frac{z^2}{z^2+1} - \frac{1}{2}\frac{z}{z^2+1} \\
&= \frac{1}{2}\left[1 - \cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right] \\
\therefore y(k) &= \frac{1}{2}\left[1 - \cos\frac{k\pi}{2} - \sin\frac{k\pi}{2}\right]
\end{aligned}
$$

# Unit-4

## Fourier Transforms

The Fourier transform pair for $f(x)$ is:

The Fourier transform of $f(x)$ is

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

The Inverse Fourier transform is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

Note:

Parseval's identity is

$$\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

1. Show that the Fourier transform of $f(x) = \begin{cases} a^2 - x^2, & |x| < a; \\ 0, & |x| > a > 0 \end{cases}$ is $2\sqrt{\frac{2}{\pi}}\left(\frac{\sin as - as\cos as}{s^3}\right)$. Hence deduce that $\int_0^\infty \frac{\sin t - t\cos t}{t^3} dt = \frac{\pi}{4}$. Using parseval's identity, show that $\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}$.

**Solution:**

The Fourier transform of $f(x)$ is

$$\begin{aligned}
F(s) &= F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2)(\cos sx + i \sin sx) dx \\
&= \frac{1}{\sqrt{2\pi}} \left[ \int_{-a}^{a} (a^2 - x^2) \cos sx dx + i \int_{-a}^{a} (a^2 - x^2) \sin sx dx \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[ 2 \int_{0}^{a} (a^2 - x^2) \cos sx dx + 0 \right] \\
&= \frac{2}{\sqrt{2\pi}} \left[ (a^2 - x^2) \frac{\sin sx}{s} - (-2x) \left( \frac{-\cos sx}{s^2} \right) + (-2) \left( \frac{-\sin sx}{s^3} \right) \right]_0^a \\
&= \frac{2}{\sqrt{2\pi}} \left[ (a^2 - x^2) \frac{\sin sx}{s} - 2x \left( \frac{\cos sx}{s^2} \right) + 2 \left( \frac{\sin sx}{s^3} \right) \right]_0^a \\
&= \sqrt{\frac{2}{\pi}} \left[ -2a \frac{\cos sa}{s^2} + 2 \frac{\sin sa}{s^3} \right] \\
&= 2\sqrt{\frac{2}{\pi}} \left[ \frac{-as\cos as + \sin as}{s^3} \right]
\end{aligned}$$

$$= \quad 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin as - as\cos as}{s^3}\right)$$

The Inverse Fourier transform is

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin as - as\cos as}{s^3}\right)(\cos sx - i\sin sx)ds \\
&= \frac{2}{\pi}\left[\int_{-\infty}^{\infty}\left(\frac{\sin as - as\cos as}{s^3}\right)\cos sxds - i\int_{-\infty}^{\infty}\left(\frac{\sin as - as\cos as}{s^3}\right)\sin sxds\right] \\
&= \frac{2}{\pi}\left[2\int_{0}^{\infty}\left(\frac{\sin as - as\cos as}{s^3}\right)\cos sxds + 0\right] \\
f(x) &= \frac{4}{\pi}\int_{0}^{\infty}\left(\frac{\sin as - as\cos as}{s^3}\right)\cos sxds
\end{aligned}$$

$$\int_{0}^{\infty}\left(\frac{\sin as - as\cos as}{s^3}\right)\cos sxds = \frac{\pi}{4}f(x)$$

Put $x = 0$ and $a = 1$

$$\begin{aligned}
\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)ds &= \frac{\pi}{4}f(0) \\
&= \frac{\pi}{4}(1) \\
\Rightarrow \int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)ds &= \frac{\pi}{4}
\end{aligned}$$

Put $s = t \quad \Rightarrow ds = dt$

$$\int_{0}^{\infty}\left(\frac{\sin t - t\cos t}{t^3}\right)dt = \frac{\pi}{4}$$

Using Parseval's identity

$$\begin{aligned}
\int_{-\infty}^{\infty}|F(s)|^2ds &= \int_{-\infty}^{\infty}|f(x)|^2dx \\
\int_{-\infty}^{\infty}4\frac{2}{\pi}\left(\frac{\sin as - as\cos as}{s^3}\right)^2ds &= \int_{-a}^{a}\left(a^2 - x^2\right)^2dx \\
2\frac{8}{\pi}\int_{0}^{\infty}\left(\frac{\sin as - as\cos as}{s^3}\right)^2ds &= 2\int_{0}^{a}\left(a^4 - 2a^2x^2 + x^4\right)dx
\end{aligned}$$

$$\frac{8}{\pi}\int_0^\infty \left(\frac{\sin as - as\cos as}{s^3}\right)^2 ds = \left(a^4x - 2a^2\frac{x^3}{3} + \frac{x^5}{5}\right)_0^a$$

$$\int_0^\infty \left(\frac{\sin as - as\cos as}{s^3}\right)^2 ds = \frac{\pi}{8}\left(a^5 - \frac{2a^5}{3} + \frac{a^5}{5}\right)$$

$$= \frac{a^5\pi}{8}\left(1 - \frac{2}{3} + \frac{1}{5}\right)$$

$$= \frac{a^5\pi}{8}\left(\frac{15-10+3}{15}\right)$$

$$= \frac{a^5\pi}{15}$$

Put $a = 1$

$$\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{\pi}{15}$$

Put $s = t \qquad \Rightarrow ds = dt$

$$\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}$$

2. Find the Fourier transform of $f(x) = \begin{cases} a^2 - x^2, & |x| \le a; \\ 0, & |x| > a > 0 \end{cases}$ Hence prove that $\int_0^\infty \frac{\sin s - s\cos s}{s^3}\cos\frac{s}{2}ds = \frac{3\pi}{16}$.

**Solution:**

The Fourier transform of $f(x)$ is

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a^2 - x^2)(\cos sx + i\sin sx)dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a}(a^2 - x^2)\cos sx dx + i\int_{-a}^{a}(a^2 - x^2)\sin sx dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[2\int_0^a (a^2 - x^2)\cos sx dx + 0\right]$$

$$= \frac{2}{\sqrt{2\pi}}\left[(a^2 - x^2)\frac{\sin sx}{s} - (-2x)\left(\frac{-\cos sx}{s^2}\right) + (-2)\left(\frac{-\sin sx}{s^3}\right)\right]_0^a$$

$$= \frac{2}{\sqrt{2\pi}}\left[(a^2 - x^2)\frac{\sin sx}{s} - 2x\left(\frac{\cos sx}{s^2}\right) + 2\left(\frac{\sin sx}{s^3}\right)\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left[-2a\frac{\cos sa}{s^2} + 2\frac{\sin sa}{s^3}\right]$$

$$= 2\sqrt{\frac{2}{\pi}}\left[\frac{-as\cos as + \sin as}{s^3}\right]$$

$$= 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin as - as\cos as}{s^3}\right)$$

The Inverse Fourier transform is

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin as - as\cos as}{s^3}\right)(\cos sx - i\sin sx)ds \\
&= \frac{2}{\pi}\left[\int_{-\infty}^{\infty}\left(\frac{\sin as - as\cos as}{s^3}\right)\cos sxds - i\int_{-\infty}^{\infty}\left(\frac{\sin as - as\cos as}{s^3}\right)\sin sxds\right] \\
&= \frac{2}{\pi}\left[2\int_{0}^{\infty}\left(\frac{\sin as - as\cos as}{s^3}\right)\cos sxds + 0\right] \\
f(x) &= \frac{4}{\pi}\int_{0}^{\infty}\left(\frac{\sin as - as\cos as}{s^3}\right)\cos sxds
\end{aligned}$$

$$\int_{0}^{\infty}\left(\frac{\sin as - as\cos as}{s^3}\right)\cos sxds = \frac{\pi}{4}f(x)$$

Put $x = \frac{1}{2}$ and $a = 1$

$$\begin{aligned}
\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos\frac{s}{2}ds &= \frac{\pi}{4}f\left(\frac{1}{2}\right) \\
&= \frac{\pi}{4}\frac{3}{4} \\
\Rightarrow \int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos\frac{s}{2}ds &= \frac{3\pi}{16}
\end{aligned}$$

3. Show that the Fourier transform of $f(x) = \begin{cases} 1-x^2, & |x| \le 1; \\ 0, & |x| > 1 \end{cases}$ is $2\sqrt{\frac{2}{\pi}}\left(\frac{\sin s - s\cos s}{s^3}\right)$. Hence deduce that $\int_0^\infty \frac{\sin t - t\cos t}{t^3}dt = \frac{\pi}{4}$. Using parseval's identity, show that $\int_0^\infty\left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}$.

**Solution:**

The Fourier transform of $f(x)$ is

$$\begin{aligned}
F(s) &= F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-x^2)(\cos sx + i\sin sx)dx \\
&= \frac{1}{\sqrt{2\pi}}\left[\int_{-1}^{1}(1-x^2)\cos sxdx + i\int_{-1}^{1}(1-x^2)\sin sxdx\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}}\left[2\int_0^1 (1-x^2)\cos sx dx + 0\right] \\
&= \frac{2}{\sqrt{2\pi}}\left[(1-x^2)\frac{\sin sx}{s} - (-2x)\left(\frac{-\cos sx}{s^2}\right) + (-2)\left(\frac{-\sin sx}{s^3}\right)\right]_0^1 \\
&= \frac{2}{\sqrt{2\pi}}\left[(1-x^2)\frac{\sin sx}{s} - 2x\left(\frac{\cos sx}{s^2}\right) + 2\left(\frac{\sin sx}{s^3}\right)\right]_0^1 \\
&= \sqrt{\frac{2}{\pi}}\left[-2\frac{\cos s}{s^2} + 2\frac{\sin s}{s^3}\right] \\
&= 2\sqrt{\frac{2}{\pi}}\left[\frac{-s\cos s + \sin s}{s^3}\right] \\
&= 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin s - s\cos s}{s^3}\right)
\end{aligned}
$$

The Inverse Fourier transform is

$$
\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx} ds \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin s - s\cos s}{s^3}\right)(\cos sx - i\sin sx) ds \\
&= \frac{2}{\pi}\left[\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sx ds - i\int_{-\infty}^{\infty}\left(\frac{\sin as - as\cos as}{s^3}\right)\sin sx ds\right] \\
&= \frac{2}{\pi}\left[2\int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sx ds + 0\right] \\
f(x) &= \frac{4}{\pi}\int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sx ds
\end{aligned}
$$

$$
\int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sx ds = \frac{\pi}{4}f(x)
$$

Put $x = 0$

$$
\begin{aligned}
\int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right) ds &= \frac{\pi}{4}f(0) \\
&= \frac{\pi}{4}(1) \\
\Rightarrow \int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right) ds &= \frac{\pi}{4}
\end{aligned}
$$

Put $s = t$ $\qquad \Rightarrow ds = dt$

$$
\int_0^{\infty}\left(\frac{\sin t - t\cos t}{t^3}\right) dt = \frac{\pi}{4}
$$

Using Parseval's identity

$$\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$\int_{-\infty}^{\infty} 4\frac{2}{\pi}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \int_{-1}^{1} \left(1-x^2\right)^2 dx$$

$$2\frac{8}{\pi}\int_{0}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\int_{0}^{1} \left(1-2x^2+x^4\right) dx$$

$$\frac{8}{\pi}\int_{0}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \left(x - 2\frac{x^3}{3} + \frac{x^5}{5}\right)_0^1$$

$$\int_{0}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{\pi}{8}\left(1-\frac{2}{3}+\frac{1}{5}\right)$$

$$= \frac{\pi}{8}\left(\frac{15-10+3}{15}\right)$$

$$= \frac{\pi}{15}$$

$$\int_{0}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{\pi}{15}$$

Put $s = t$ $\Rightarrow ds = dt$

$$\int_{0}^{\infty} \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}$$

4. Find the Fourier transform of $f(x) = \begin{cases} 1-x^2, & |x| \le 1; \\ 0, & |x| > 1 \end{cases}$ Hence prove that $\int_0^\infty \frac{\sin s - s\cos s}{s^3}\cos\frac{s}{2}ds = \frac{3\pi}{16}$.

**Solution:**

The Fourier transform of $f(x)$ is

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-x^2)(\cos sx + i\sin sx)dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-1}^{1}(1-x^2)\cos sx dx + i\int_{-1}^{1}(1-x^2)\sin sx dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[2\int_{0}^{1}(1-x^2)\cos sx dx + 0\right]$$

$$= \frac{2}{\sqrt{2\pi}}\left[(1-x^2)\frac{\sin sx}{s} - (-2x)\left(\frac{-\cos sx}{s^2}\right) + (-2)\left(\frac{-\sin sx}{s^3}\right)\right]_0^1$$

$$= \frac{2}{\sqrt{2\pi}}\left[(1-x^2)\frac{\sin sx}{s} - 2x\left(\frac{\cos sx}{s^2}\right) + 2\left(\frac{\sin sx}{s^3}\right)\right]_0^1$$

$$= \sqrt{\frac{2}{\pi}}\left[-2\frac{\cos s}{s^2} + 2\frac{\sin s}{s^3}\right]$$

$$= 2\sqrt{\frac{2}{\pi}}\left[\frac{-s\cos s+\sin s}{s^3}\right]$$
$$= 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin s - s\cos s}{s^3}\right)$$

The Inverse Fourier transform is

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin s - s\cos s}{s^3}\right)(\cos sx - i\sin sx)ds \\
&= \frac{2}{\pi}\left[\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sxds - i\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\sin sxds\right] \\
&= \frac{2}{\pi}\left[2\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sxds + 0\right] \\
f(x) &= \frac{4}{\pi}\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sxds \\
\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sxds &= \frac{\pi}{4}f(x)
\end{aligned}$$

Put $x = \frac{1}{2}$

$$\begin{aligned}
\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos\frac{s}{2}ds &= \frac{\pi}{4}f\left(\frac{1}{2}\right) \\
&= \frac{\pi}{4}\frac{3}{4} \\
\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos\frac{s}{2}ds &= \frac{3\pi}{16}
\end{aligned}$$

5. Find the Fourier transform of $f(x) = \begin{cases} a-|x|, & |x|<a; \\ 0, & |x|>a>0 \end{cases}$ and deduce the value of $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt$ and $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt$.

**Solution:**

The Fourier transform of $f(x)$ is

$$\begin{aligned}
F(s) &= F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a-|x|)(\cos sx + i\sin sx)dx \\
&= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a}(a-|x|)\cos sxdx + i\int_{-a}^{a}(a-|x|)\sin sxdx\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}}\left[2\int_0^a (a-x)\cos sx dx + 0\right] \\
&= \frac{2}{\sqrt{2\pi}}\left[(a-x)\frac{\sin sx}{s} - (-1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^a \\
&= \frac{2}{\sqrt{2\pi}}\left[(a-x)\frac{\sin sx}{s} - \left(\frac{\cos sx}{s^2}\right)\right]_0^a \\
&= \sqrt{\frac{2}{\pi}}\left[-\frac{\cos sa}{s^2} - \left(-\frac{1}{s^2}\right)\right] \\
&= \sqrt{\frac{2}{\pi}}\left[\frac{1-\cos as}{s^2}\right] \\
&= \sqrt{\frac{2}{\pi}}\left(\frac{2\sin^2\frac{as}{2}}{s^2}\right) \\
&= 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin^2\frac{as}{2}}{s^2}\right)
\end{aligned}
$$

The Inverse Fourier transform is

$$
\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin^2\frac{as}{2}}{s^2}\right)(\cos sx - i\sin sx)ds \\
&= \frac{2}{\pi}\left[\int_{-\infty}^{\infty}\left(\frac{\sin^2\frac{as}{2}}{s^2}\right)\cos sxds - i\int_{-\infty}^{\infty}\left(\frac{\sin^2\frac{as}{2}}{s^2}\right)\sin sxds\right] \\
&= \frac{2}{\pi}\left[2\int_0^{\infty}\left(\frac{\sin^2\frac{as}{2}}{s^2}\right)\cos sxds + 0\right] \\
f(x) &= \frac{4}{\pi}\int_0^{\infty}\left(\frac{\sin^2\frac{as}{2}}{s^2}\right)\cos sxds
\end{aligned}
$$

$$
\int_0^{\infty}\left(\frac{\sin^2\frac{as}{2}}{s^2}\right)\cos sxds = \frac{\pi}{4}f(x)
$$

Put $x = 0$ and $a = 1$

$$
\begin{aligned}
\int_0^{\infty}\left(\frac{\sin^2\frac{s}{2}}{s^2}\right)ds &= \frac{\pi}{4}f(0) \\
&= \frac{\pi}{4}(1) \\
\Rightarrow \int_0^{\infty}\left(\frac{\sin^2\frac{s}{2}}{s^2}\right)ds &= \frac{\pi}{4}
\end{aligned}
$$

Put $\frac{s}{2} = t \quad \Rightarrow s = 2t \quad \Rightarrow ds = 2dt$

$$\begin{aligned}
\int_0^\infty \left(\frac{\sin^2 t}{4t^2}\right) 2dt &= \frac{\pi}{4} \\
\frac{1}{2}\int_0^\infty \left(\frac{\sin^2 t}{t^2}\right) dt &= \frac{\pi}{4} \\
\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt &= \frac{\pi}{2}
\end{aligned}$$

Using Parseval's identity

$$\begin{aligned}
\int_{-\infty}^{\infty} |F(s)|^2 ds &= \int_{-\infty}^{\infty} |f(x)|^2 dx \\
\int_{-\infty}^{\infty} 4\frac{2}{\pi}\left(\frac{\sin^2 \frac{as}{2}}{s^2}\right)^2 ds &= \int_{-a}^{a} (a-|x|)^2 dx \\
2\frac{8}{\pi}\int_0^\infty \left(\frac{\sin^2 \frac{as}{2}}{s^2}\right)^2 ds &= 2\int_0^a (a-x)^2 dx \\
\frac{8}{\pi}\int_0^\infty \left(\frac{\sin^2 \frac{as}{2}}{s^2}\right)^2 ds &= \int_0^a (a^2-2ax+x^2) dx \\
\frac{8}{\pi}\int_0^\infty \left(\frac{\sin^2 \frac{as}{2}}{s^2}\right)^2 ds &= \left(a^2x - 2a\frac{x^2}{2} + \frac{x^3}{3}\right)_0^a \\
\int_0^\infty \left(\frac{\sin^2 \frac{as}{2}}{s^2}\right)^2 ds &= \frac{\pi}{8}\left(a^3 - a^3 + \frac{a^3}{3}\right) \\
&= \frac{\pi}{8}\left(\frac{a^3}{3}\right) \\
&= \frac{a^3\pi}{24}
\end{aligned}$$

Put $a = 1$

$$\int_0^\infty \left(\frac{\sin^4 \frac{s}{2}}{s^4}\right) ds = \frac{\pi}{24}$$

Put $\frac{s}{2} = t \quad \Rightarrow s = 2t \quad \Rightarrow ds = 2dt$

$$\begin{aligned}
\int_0^\infty \left(\frac{\sin^4 t}{16t^4}\right) 2dt &= \frac{\pi}{24} \\
\frac{1}{8}\int_0^\infty \left(\frac{\sin^4 t}{t^4}\right) dt &= \frac{\pi}{24} \\
\int_0^\infty \left(\frac{\sin^4 t}{t^4}\right) dt &= \frac{\pi}{3}
\end{aligned}$$

6. Find the Fourier transform of $f(x) = \begin{cases} 1-|x|, & |x|<1; \\ 0, & |x|>1 \end{cases}$ and deduce the value of $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt$ and $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt$.

**Solution:**

The Fourier transform of $f(x)$ is

$$\begin{aligned}
F(s) &= F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)(\cos sx + i\sin sx)dx \\
&= \frac{1}{\sqrt{2\pi}}\left[\int_{-1}^{1}(1-|x|)\cos sxdx + i\int_{-1}^{1}(1-|x|)\sin sxdx\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[2\int_{0}^{1}(1-x)\cos sxdx + 0\right] \\
&= \frac{2}{\sqrt{2\pi}}\left[(1-x)\frac{\sin sx}{s} - (-1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^1 \\
&= \frac{2}{\sqrt{2\pi}}\left[(1-x)\frac{\sin sx}{s} - \left(\frac{\cos sx}{s^2}\right)\right]_0^1 \\
&= \sqrt{\frac{2}{\pi}}\left[-\frac{\cos s}{s^2} - \left(-\frac{1}{s^2}\right)\right] \\
&= \sqrt{\frac{2}{\pi}}\left[\frac{1-\cos s}{s^2}\right] \\
&= \sqrt{\frac{2}{\pi}}\left(\frac{2\sin^2\frac{s}{2}}{s^2}\right) \\
&= 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin^2\frac{s}{2}}{s^2}\right)
\end{aligned}$$

The Inverse Fourier transform is

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin^2\frac{s}{2}}{s^2}\right)(\cos sx - i\sin sx)ds \\
&= \frac{2}{\pi}\left[\int_{-\infty}^{\infty}\left(\frac{\sin^2\frac{s}{2}}{s^2}\right)\cos sxds - i\int_{-\infty}^{\infty}\left(\frac{\sin^2\frac{s}{2}}{s^2}\right)\sin sxds\right] \\
&= \frac{2}{\pi}\left[2\int_{0}^{\infty}\left(\frac{\sin^2\frac{s}{2}}{s^2}\right)\cos sxds + 0\right] \\
f(x) &= \frac{4}{\pi}\int_{0}^{\infty}\left(\frac{\sin^2\frac{s}{2}}{s^2}\right)\cos sxds
\end{aligned}$$

$$\int_0^\infty \left(\frac{\sin^2 \frac{s}{2}}{s^2}\right) \cos sx ds = \frac{\pi}{4} f(x)$$

Put $x = 0$

$$\begin{aligned}
\int_0^\infty \left(\frac{\sin^2 \frac{s}{2}}{s^2}\right) ds &= \frac{\pi}{4} f(0) \\
&= \frac{\pi}{4}(1) \\
\Rightarrow \int_0^\infty \left(\frac{\sin^2 \frac{s}{2}}{s^2}\right) ds &= \frac{\pi}{4}
\end{aligned}$$

Put $\frac{s}{2} = t \qquad \Rightarrow s = 2t \qquad \Rightarrow ds = 2dt$

$$\begin{aligned}
\int_0^\infty \left(\frac{\sin^2 t}{4t^2}\right) 2dt &= \frac{\pi}{4} \\
\frac{1}{2}\int_0^\infty \left(\frac{\sin^2 t}{t^2}\right) dt &= \frac{\pi}{4} \\
\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt &= \frac{\pi}{2}
\end{aligned}$$

Using Parseval's identity

$$\begin{aligned}
\int_{-\infty}^\infty |F(s)|^2 ds &= \int_{-\infty}^\infty |f(x)|^2 dx \\
\int_{-\infty}^\infty 4\frac{2}{\pi} \left(\frac{\sin^2 \frac{s}{2}}{s^2}\right)^2 ds &= \int_{-1}^1 (1-|x|)^2 dx \\
2\frac{8}{\pi}\int_0^\infty \left(\frac{\sin^2 \frac{s}{2}}{s^2}\right)^2 ds &= 2\int_0^1 (1-x)^2 dx \\
\frac{8}{\pi}\int_0^\infty \left(\frac{\sin^2 \frac{s}{2}}{s^2}\right)^2 ds &= \int_0^1 (1-2x+x^2) dx \\
\frac{8}{\pi}\int_0^\infty \left(\frac{\sin^2 \frac{s}{2}}{s^2}\right)^2 ds &= \left(a^2x - 2a\frac{x^2}{2} + \frac{x^3}{3}\right)_0^1 \\
\int_0^\infty \left(\frac{\sin^2 \frac{s}{2}}{s^2}\right)^2 ds &= \frac{\pi}{8}\left(1 - 1 + \frac{1}{3}\right) \\
&= \frac{\pi}{8}\left(\frac{1}{3}\right) \\
&= \frac{\pi}{24} \\
\int_0^\infty \left(\frac{\sin^4 \frac{s}{2}}{s^4}\right) ds &= \frac{\pi}{24}
\end{aligned}$$

Put $\frac{s}{2} = t \qquad \Rightarrow s = 2t \qquad \Rightarrow ds = 2dt$

$$\begin{aligned}
\int_0^\infty \left(\frac{\sin^4 t}{16t^4}\right) 2dt &= \frac{\pi}{24} \\
\frac{1}{8}\int_0^\infty \left(\frac{\sin^4 t}{t^4}\right) dt &= \frac{\pi}{24} \\
\int_0^\infty \left(\frac{\sin^4 t}{t^4}\right) dt &= \frac{\pi}{3}
\end{aligned}$$

7. Find the Fourier transform of $f(x) = \begin{cases} 1, & |x| < a; \\ 0, & |x| > a \end{cases}$ and deduce the value of $\int_0^\infty \frac{\sin t}{t} dt$ and $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt$.

**Solution:**

The Fourier transform of $f(x)$ is

$$\begin{aligned}
F(s) &= F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (1)(\cos sx + i \sin sx) dx \\
&= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{a} \cos sx dx + i \int_{-a}^{a} \sin sx dx\right] \\
&= \frac{1}{\sqrt{2\pi}} \left[2\int_0^a \cos sx dx + 0\right] \\
&= \frac{2}{\sqrt{2\pi}} \left[\frac{\sin sx}{s}\right]_0^a \\
&= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sa}{s}\right]
\end{aligned}$$

The Inverse Fourier transform is

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left[\frac{\sin sa}{s}\right] (\cos sx - i \sin sx) ds \\
&= \frac{1}{\pi} \left[\int_{-\infty}^{\infty} \left(\frac{\sin sa}{s}\right) \cos sx ds - i \int_{-\infty}^{\infty} \left(\frac{\sin sa}{s}\right) \sin sx ds\right] \\
&= \frac{1}{\pi} \left[2\int_0^\infty \left(\frac{\sin sa}{s}\right) \cos sx ds - 0\right] \\
f(x) &= \frac{2}{\pi} \int_0^\infty \left(\frac{\sin sa}{s}\right) \cos sx ds
\end{aligned}$$

$$\int_0^\infty \left(\frac{\sin sa}{s}\right)\cos sx ds = \frac{\pi}{2}f(x)$$

Put $x = 0$ and $a = 1$

$$\int_0^\infty \left(\frac{\sin s}{s}\right) ds = \frac{\pi}{2}f(0)$$

Put $s = x \quad \Rightarrow ds = dx$

$$\Rightarrow \int_0^\infty \left(\frac{\sin x}{x}\right) dx = \frac{\pi}{2}$$

Using Parseval's identity

$$\begin{aligned}
\int_{-\infty}^{\infty} |F(s)|^2 ds &= \int_{-\infty}^{\infty} |f(x)|^2 dx \\
\int_{-\infty}^{\infty} \frac{2}{\pi}\left(\frac{\sin as}{s}\right)^2 ds &= \int_{-a}^{a} (1)^2 dx \\
\frac{2}{\pi}2\int_0^\infty \left(\frac{\sin as}{s}\right)^2 ds &= 2\int_0^a dx \\
\frac{2}{\pi}\int_0^\infty \left(\frac{\sin as}{s}\right)^2 ds &= [x]_0^a \\
&= a \\
\int_0^\infty \left(\frac{\sin as}{s}\right)^2 ds &= \frac{\pi}{2}a
\end{aligned}$$

Put $a = 1$

$$\int_0^\infty \left(\frac{\sin s}{s}\right)^2 ds = \frac{\pi}{2}$$

Put $s = x \quad \Rightarrow ds = dx$

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$$

8. Find the Fourier transform of $f(x) = \begin{cases} 1, & |x| < 2; \\ 0, & |x| > 2 \end{cases}$ and hence evaluate $\int_0^\infty \frac{\sin x}{x} dx$ and $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx$.

**Note:**

$$
\begin{aligned}
1. \int e^{ax}\cos bx dx &= \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\\
2. \int e^{ax}\sin bx dx &= \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\\
3. \int_0^\infty e^{-ax}\cos sx dx &= \left[\frac{e^{-ax}}{(-a)^2+s^2}(-a\cos sx + s\sin sx)\right]_0^\infty\\
&= \left[0-\frac{1}{a^2+s^2}(-a)\right]\\
&= \frac{a}{s^2+a^2}\\
4. \int_0^\infty e^{-ax}\sin sx dx &= \left[\frac{e^{-ax}}{(-a)^2+s^2}(-a\sin sx - s\cos sx)\right]_0^\infty\\
&= \left[0-\frac{1}{a^2+s^2}(-s)\right]\\
&= \frac{s}{s^2+a^2}
\end{aligned}
$$

9. Find the Fourier Transform of $e^{-a|x|}, a > 0$.Show that $\int_0^\infty \frac{\cos sx}{(a^2+s^2)}ds = \frac{\pi}{2}e^{-a|x|}$.

Hence deduce that $F[xe^{-a|x|}] = i\sqrt{\frac{2}{\pi}}\frac{2as}{(a^2+s^2)^2}$.

**Solution:**

The Fourier transform of $f(x)$ is

$$
\begin{aligned}
F(s) &= F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}(\cos sx + i\sin sx)dx\\
&= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} e^{-a|x|}\cos sx dx + i\int_{-\infty}^{\infty} e^{-a|x|}\sin sx dx\right]\\
&= \frac{1}{\sqrt{2\pi}}\left[2\int_0^{\infty} e^{-ax}\cos sx dx + 0\right]\\
&= \sqrt{\frac{2}{\pi}}\frac{a}{(a^2+s^2)}
\end{aligned}
$$

The Inverse Fourier transform is

$$
\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\frac{a}{(a^2+s^2)}(\cos sx - i\sin sx)ds\\
&= \frac{1}{\pi}\left[\int_{-\infty}^{\infty}\frac{a}{(a^2+s^2)}\cos sx ds - i\int_{-\infty}^{\infty}\frac{a}{(a^2+s^2)}\sin sx ds\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{a}{\pi}\left[2\int_0^\infty \frac{1}{(a^2+s^2)}\cos sx ds + 0\right] \\
f(x) &= \frac{2a}{\pi}\int_0^\infty \frac{\cos sx}{(a^2+s^2)} ds \\
\int_0^\infty \frac{\cos sx}{(a^2+s^2)} ds &= \frac{\pi}{2a} f(x) \\
\int_0^\infty \frac{\cos sx}{(a^2+s^2)} ds &= \frac{\pi}{2a} e^{-a|x|}
\end{aligned}
$$

W.K.T $F[xf(x)] = -i\frac{d}{ds}F[f(x)]$

$$
\begin{aligned}
F[xe^{-a|x|}] &= -i\frac{d}{ds}\sqrt{\frac{2}{\pi}}\frac{a}{(a^2+s^2)} \\
&= -ia\sqrt{\frac{2}{\pi}}\left[\frac{-1}{(a^2+s^2)^2}\right]2s \\
&= i\sqrt{\frac{2}{\pi}}\frac{2as}{(a^2+s^2)^2}
\end{aligned}
$$

10. Find the Fourier Transform of $e^{-|x|}$.Find the value of $F[e^{-|x|}\cos 2x]$.

**Solution:**

The Fourier transform of $f(x)$ is

$$
\begin{aligned}
F(s) = F[f(x)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|x|}(\cos sx + i\sin sx)dx \\
&= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} e^{-|x|}\cos sxdx + i\int_{-\infty}^{\infty} e^{-|x|}\sin sxdx\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[2\int_0^\infty e^{-x}\cos sxdx + 0\right] \\
&= \sqrt{\frac{2}{\pi}}\frac{1}{(1+s^2)}
\end{aligned}
$$

W.K.T $F[f(x)\cos ax] = \frac{1}{2}[F(s+a)+F(s-a)], a = 2$

$$
\begin{aligned}
F(s+2) &= \sqrt{\frac{2}{\pi}}\frac{1}{(1+(s+2)^2)} \\
F(s-2) &= \sqrt{\frac{2}{\pi}}\frac{1}{(1+(s-2)^2)} \\
F[e^{-|x|}\cos 2x] &= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\frac{1}{(1+(s+2)^2)} + \sqrt{\frac{2}{\pi}}\frac{1}{(1+(s-2)^2)}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{1}{s^2+4s+5}+\frac{1}{s^2-4s+5}\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{s^2-4s+5+s^2+4s+5}{(s^2+4s+5)(s^2-4s+5)}\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{2s^2+10}{(s^2+5)^2-(4s)^2}\right] \\
&= \frac{2}{\sqrt{2\pi}}\left[\frac{s^2+5}{s^4+10s^2+25-16s^2}\right] \\
&= \sqrt{\frac{2}{\pi}}\left[\frac{s^2+5}{s^4-6s^2+25}\right]
\end{aligned}
$$

11. Find the Fourier transform of $f(x) = \begin{cases} x, & |x| < a; \\ 0, & |x| > a \end{cases}$

**Solution:**

The Fourier transform of $f(x)$ is

$$
\begin{aligned}
F(s) &= F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} x(\cos sx + i\sin sx)dx \\
&= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a} x\cos sx dx + i\int_{-a}^{a} x\sin sx dx\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[0 + 2i\int_{0}^{a} x\sin sx dx\right] \\
&= \frac{2i}{\sqrt{2\pi}}\left[x\left(\frac{-\cos sx}{s}\right) - (1)\left(\frac{-\sin sx}{s^2}\right)\right]_0^a \\
&= i\sqrt{\frac{2}{\pi}}\left[-\frac{x\cos sx}{s} + \frac{\sin sx}{s^2}\right]_0^a \\
&= i\sqrt{\frac{2}{\pi}}\left[-\frac{a\cos as}{s} + \frac{\sin as}{s^2}\right] \\
&= i\sqrt{\frac{2}{\pi}}\left[\frac{\sin as - as\cos as}{s^2}\right]
\end{aligned}
$$

## Fourier cosine and sine transforms

The Fourier Cosine transform of $f(x)$ is

$$F_c[s] = F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx dx$$

The Inverse Fourier Cosine transform is

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F[s]\cos sx ds$$

The Fourier Sine transform of $f(x)$ is

$$F_s[s] = F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx dx$$

The Inverse Fourier Sine transform is

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F[s]\sin sx ds$$

## Self Reciprocal

12. Find the Fourier Cosine transform of $e^{-a^2x^2}$.Hence show that $e^{-\frac{x^2}{2}}$ is self reciprocal under Fourier Cosine transform and find $F_s\left[xe^{-\frac{x2}{2}}\right]$.

**Solution:**

The Fourier Cosine transform of $f(x)$ is

$$F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx dx$$

$$F_c[e^{-a^2x^2}] = \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_{-\infty}^\infty f(x) \text{ R.p of } e^{isx}dx$$

$$= \text{R.p of } \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x)e^{isx}dx$$

$$= \text{R.p of } \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-a^2x^2}e^{isx}dx$$

$$= \text{R.p of } \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-a^2x^2}e^{isx}e^{-\frac{s^2}{4a^2}}e^{\frac{s^2}{4a^2}}dx$$

$$= \text{R.p of } e^{-\frac{s^2}{4a^2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-\left(a^2x^2-isx-\frac{s^2}{4a^2}\right)}dx$$

$$= \text{R.p of } e^{-\frac{s^2}{4a^2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-\left(a^2x^2-isx+\frac{i^2s^2}{2^2a^2}\right)}dx$$

$$= \text{R.p of } e^{-\frac{s^2}{4a^2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-\left(a^2x^2-\frac{2axis}{2a}+\frac{i^2s^2}{2^2a^2}\right)}dx$$

$$= \text{R.p of } e^{-\frac{s^2}{4a^2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax-\frac{is}{2a}\right)^2} dx$$

Put $u = ax - \frac{is}{2a}$ $\Rightarrow du = adx$

if $x = -\infty$ $\Rightarrow u = -\infty$

if $x = \infty$ $\Rightarrow u = \infty$

$$F_c[e^{-a^2x^2}] = \text{R.p of } e^{-\frac{s^2}{4a^2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} \frac{du}{a}$$

$$= \text{R.p of } e^{-\frac{s^2}{4a^2}} \frac{1}{a\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} du$$

$$= \text{R.p of } e^{-\frac{s^2}{4a^2}} \frac{1}{a\sqrt{2\pi}} \sqrt{\pi}$$

$$F_c[e^{-a^2x^2}] = \frac{e^{-\frac{s^2}{4a^2}}}{a\sqrt{2}}$$

Now,put $a = \frac{1}{\sqrt{2}}$

$$F_c[e^{\frac{-x^2}{2}}] = \frac{e^{-\frac{s^2}{4(\frac{1}{2})}}}{\frac{1}{\sqrt{2}}\sqrt{2}}$$

$$F_c[e^{\frac{-x^2}{2}}] = e^{-\frac{s^2}{2}}$$

W.k.t $F_s[xf(x)] = -\frac{d}{ds}[F_c(f(x))]$

$$F_s[xe^{\frac{x^2}{2}}] = -\frac{d}{ds}\left[e^{-\frac{s^2}{2}}\right]$$

$$= -e^{-\frac{s^2}{2}}\left(\frac{-2s}{2}\right)$$

$$= se^{-\frac{s^2}{2}}$$

13. Find the Fourier transform of $e^{-a^2x^2}$.Hence show that $e^{-\frac{x^2}{2}}$ is self reciprocal under Fourier transform and find $F\left[xe^{-\frac{x^2}{2}}\right]$.

**Solution:**

The Fourier transform of $f(x)$ is

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2}e^{isx}dx$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} e^{-\frac{s^2}{4a^2}} e^{\frac{s^2}{4a^2}} dx \\
&= e^{-\frac{s^2}{4a^2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(a^2x^2 - isx - \frac{s^2}{4a^2}\right)} dx \\
&= e^{-\frac{s^2}{4a^2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(a^2x^2 - isx + \frac{i^2s^2}{2^2a^2}\right)} \\
&= e^{-\frac{s^2}{4a^2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(a^2x^2 - \frac{2axis}{2a} + \frac{i^2s^2}{2^2a^2}\right)} dx \\
&= e^{-\frac{s^2}{4a^2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} dx
\end{aligned}
$$

Put $u = ax - \dfrac{is}{2a}$ $\qquad \Rightarrow du = adx$

if $x = -\infty$ $\qquad \Rightarrow u = -\infty$

if $x = \infty$ $\qquad \Rightarrow u = \infty$

$$
\begin{aligned}
F[f(x)] &= e^{-\frac{s^2}{4a^2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} \frac{du}{a} \\
&= e^{-\frac{s^2}{4a^2}} \frac{1}{a\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} du \\
&= e^{-\frac{s^2}{4a^2}} \frac{1}{a\sqrt{2\pi}} \sqrt{\pi} \\
F[e^{-a^2x^2}] &= \frac{e^{-\frac{s^2}{4a^2}}}{a\sqrt{2}}
\end{aligned}
$$

Now,put $a = \dfrac{1}{\sqrt{2}}$

$$
\begin{aligned}
F[e^{\frac{-x^2}{2}}] &= \frac{e^{-\frac{s^2}{4(\frac{1}{2})}}}{\frac{1}{\sqrt{2}}\sqrt{2}} \\
F[e^{\frac{-x^2}{2}}] &= e^{-\frac{s^2}{2}}
\end{aligned}
$$

W.k.t $F[xf(x)] = -i\dfrac{d}{ds}[F(f(x))]$

$$
\begin{aligned}
F[xe^{\frac{x^2}{2}}] &= -i\frac{d}{ds}\left[e^{-\frac{s^2}{2}}\right] \\
&= -ie^{-\frac{s^2}{2}}\left(\frac{-2s}{2}\right) \\
&= ise^{-\frac{s^2}{2}}
\end{aligned}
$$

14. Show that $e^{-\frac{x^2}{2}}$ is self reciprocal under Fourier Cosine transform.

**Solution:**

The Fourier Cosine transform of $f(x)$ is

$$\begin{aligned}
F_c(s) &= F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx dx \\
F_c[e^{-\frac{x^2}{2}}] &= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} f(x) \text{ R.p of } e^{isx}dx \\
&= \text{R.p of } \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx \\
&= \text{R.p of } \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}e^{isx}dx \\
&= \text{R.p of } \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}e^{isx}e^{-\frac{s^2}{2}}e^{\frac{s^2}{2}}dx \\
&= \text{R.p of } e^{-\frac{s^2}{2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(x^2-2isx-s^2\right)}dx \\
&= \text{R.p of } e^{-\frac{s^2}{2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(x^2-2isx+i^2s^2\right)}dx \\
&= \text{R.p of } e^{-\frac{s^2}{2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-is)^2}dx \\
&= \text{R.p of } e^{-\frac{s^2}{2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(\frac{x-is}{\sqrt{2}}\right)^2}dx
\end{aligned}$$

Put $u = \frac{x-is}{\sqrt{2}}$ $\Rightarrow du = \frac{dx}{\sqrt{2}}$

if $x = -\infty$ $\Rightarrow u = -\infty$

if $x = \infty$ $\Rightarrow u = \infty$

$$\begin{aligned}
F[f(x)] &= \text{R.p of } e^{-\frac{s^2}{2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-u^2}\sqrt{2}du \\
&= \text{R.p of } e^{-\frac{s^2}{2}}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2}du \\
&= \text{R.p of } e^{-\frac{s^2}{2}}\frac{1}{\sqrt{\pi}}\sqrt{\pi} \\
F_c[e^{-\frac{x^2}{2}}] &= e^{-\frac{s^2}{2}}
\end{aligned}$$

15. Show that $e^{-\frac{x^2}{2}}$ is self reciprocal under Fourier transform.

Solution:

The Fourier transform of $f(x)$ is

$$\begin{aligned} F(s) &= F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{isx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{isx} e^{-\frac{s^2}{2}} e^{\frac{s^2}{2}} dx \\ &= e^{-\frac{s^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(x^2-2isx-s^2\right)} dx \\ &= e^{-\frac{s^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(x^2-2isx+i^2s^2\right)} dx \\ &= e^{-\frac{s^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-is)^2} dx \\ &= e^{-\frac{s^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x-is}{\sqrt{2}}\right)^2} dx \end{aligned}$$

Put $u = \frac{x-is}{\sqrt{2}}$ $\Rightarrow du = \frac{dx}{\sqrt{2}}$

if $x = -\infty$ $\Rightarrow u = -\infty$

if $x = \infty$ $\Rightarrow u = \infty$

$$\begin{aligned} F[f(x)] &= e^{-\frac{s^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} \sqrt{2} du \\ &= e^{-\frac{s^2}{2}} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du \\ &= e^{-\frac{s^2}{2}} \frac{1}{\sqrt{\pi}} \sqrt{\pi} \\ F[e^{-\frac{x^2}{2}}] &= e^{-\frac{s^2}{2}} \end{aligned}$$

16. Find the Fourier Cosine transform of $e^{-x^2}$.

17. Find the Fourier Cosine and Sine transform of $x^{n-1}$.Prove that $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier Cosine and Sine transforms.

**Solution:**

The Fourier Cosine transform of $f(x)$ is

$$F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos sx dx$$

$$F_c[x^{n-1}] = \sqrt{\frac{2}{\pi}}\int_0^\infty x^{n-1}\cos sx dx$$
$$= \sqrt{\frac{2}{\pi}}\frac{\Gamma n}{s^n}\cos\frac{n\pi}{2}$$

Put $n = \frac{1}{2}$

$$F_c[x^{\frac{1}{2}-1}] = \sqrt{\frac{2}{\pi}}\frac{\Gamma\frac{1}{2}}{s^{\frac{1}{2}}}\cos\frac{\pi}{4}$$
$$F_c[x^{-\frac{1}{2}}] = \sqrt{\frac{2}{\pi}}\frac{\sqrt{\pi}}{\sqrt{s}}\frac{1}{\sqrt{2}}$$
$$F_c\left[\frac{1}{\sqrt{x}}\right] = \frac{1}{\sqrt{s}}$$

The Fourier Cosine transform of $f(x)$ is

$$F_s(s) = F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx dx$$
$$F_s[x^{n-1}] = \sqrt{\frac{2}{\pi}}\int_0^\infty x^{n-1}\sin sx dx$$
$$= \sqrt{\frac{2}{\pi}}\frac{\Gamma n}{s^n}\sin\frac{n\pi}{2}$$

Put $n = \frac{1}{2}$

$$F_s[x^{\frac{1}{2}-1}] = \sqrt{\frac{2}{\pi}}\frac{\Gamma\frac{1}{2}}{s^{\frac{1}{2}}}\sin\frac{\pi}{4}$$
$$F_s[x^{-\frac{1}{2}}] = \sqrt{\frac{2}{\pi}}\frac{\sqrt{\pi}}{\sqrt{s}}\frac{1}{\sqrt{2}}$$
$$F_s\left[\frac{1}{\sqrt{x}}\right] = \frac{1}{\sqrt{s}}$$

18. Find the Fourier Cosine Transform of $f(x) = \begin{cases} x, & 0 < x < 1; \\ 2-x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$

**Solution:**

The Fourier Cosine transform of $f(x)$ is

$$F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx dx$$
$$= \sqrt{\frac{2}{\pi}}\left[\int_0^1 x\cos sx dx + \int_1^2 (2-x)\cos sx dx\right]$$
$$= \sqrt{\frac{2}{\pi}}\left[\left\{x\left(\frac{\sin sx}{s}\right) - (1)\left(\frac{-\cos sx}{s^2}\right)\right\}_0^1 + \left\{(2-x)\left(\frac{\sin sx}{s}\right) - (-1)\left(\frac{-\cos sx}{s^2}\right)\right\}_1^2\right]$$
$$= \sqrt{\frac{2}{\pi}}\left[\left\{x\left(\frac{\sin sx}{s}\right) + \frac{\cos sx}{s^2}\right\}_0^1 + \left\{(2-x)\left(\frac{\sin sx}{s}\right) - \frac{\cos sx}{s^2}\right\}_1^2\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\left\{\left(\frac{\sin s}{s}\right)+\left(\frac{\cos s}{s^2}\right)-\left(0+\frac{1}{s^2}\right)\right\}+\left\{0-\frac{\cos 2s}{s^2}-\left(\frac{\sin s}{s}-\frac{\cos s}{s^2}\right)\right\}\right]$$
$$=\sqrt{\frac{2}{\pi}}\left[\frac{2\cos s}{s^2}-\frac{1}{s^2}-\frac{\cos 2s}{s^2}\right]$$
$$=\sqrt{\frac{2}{\pi}}\frac{[2\cos s-(1+\cos 2s)]}{s^2}$$
$$=\sqrt{\frac{2}{\pi}}\frac{[2\cos s-2\cos^2 s]}{s^2}$$
$$=\sqrt{\frac{2}{\pi}}\frac{2\cos s}{s^2}[1-\cos s]$$

19. Find the Fourier Sine Transform of $f(x)=\begin{cases} x, & 0<x<1; \\ 2-x, & 1<x<2 \\ 0, & x>2 \end{cases}$

20. Find the Fourier Cosine and Sine transform of $e^{-ax}$ and Find its inversion.Also find $F_c[xe^{-ax}]$ and $F_s[xe^{-ax}]$

**Solution:**

The Fourier Cosine transform of $f(x)$ is

$$\begin{aligned} F_c(s) &= F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx dx \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx dx \\ &= \sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2} \end{aligned}$$

The Inverse Fourier Cosine transform is

$$\begin{aligned} f(x) &= \sqrt{\frac{2}{\pi}}\int_0^\infty F_c[s]\cos sx ds \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}\cos sx ds \\ &= \frac{2a}{\pi}\int_0^\infty \frac{1}{s^2+a^2}\cos sx ds \\ \int_0^\infty \frac{\cos sx}{s^2+a^2}ds &= \frac{\pi}{2a}f(x) \\ \int_0^\infty \frac{\cos sx}{s^2+a^2}ds &= \frac{\pi}{2a}e^{-ax} \end{aligned}$$

The Fourier Sine transform of $f(x)$ is

$$F_s(s) = F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx dx$$
$$= \sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2}$$

The Inverse Fourier Cosine transform is

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s)) \sin sx ds$$
$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2} \sin sx ds$$
$$= \frac{2}{\pi} \int_0^\infty \frac{s}{s^2+a^2} \sin sx ds$$
$$\int_0^\infty \frac{s \sin sx}{s^2+a^2} ds = \frac{\pi}{2} f(x)$$
$$\int_0^\infty \frac{s \sin sx}{s^2+a^2} ds = \frac{\pi}{2} e^{-ax}$$

W.k.t $F_c[xf(x)] = \frac{d}{ds}[F_s(f(x))]$

$$F_c[xe^{-ax}] = \frac{d}{ds} F_s\left[e^{-ax}\right]$$
$$= \frac{d}{ds} \sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2}$$
$$= \sqrt{\frac{2}{\pi}} \frac{(s^2+a^2)(1) - s(2s)}{(s^2+a^2)^2}$$
$$= \sqrt{\frac{2}{\pi}} \frac{a^2-s^2}{(s^2+a^2)^2}$$

W.k.t $F_s[xf(x)] = -\frac{d}{ds}[F_c(f(x))]$

$$F_s[xe^{-ax}] = -\frac{d}{ds} F_c\left[e^{-ax}\right]$$
$$= -\frac{d}{ds} \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}$$
$$= -a\sqrt{\frac{2}{\pi}} \left[\frac{-1}{(s^2+a^2)^2}\right](2s)$$
$$= \sqrt{\frac{2}{\pi}} \frac{2as}{(s^2+a^2)^2}$$

Find the Fourier Cosine and Sine transform of $e^{-x}$

## Half Range Parseval's Identity

**Formula:**

$$(i)\ \int_0^\infty F_c[f(x)]F_c[g(x)]ds = \int_0^\infty f(x)g(x)dx$$

$$(ii)\ \int_0^\infty [F_c[f(x)]]^2\, ds = \int_0^\infty [f(x)]^2\, dx$$

$$(iii)\ \int_0^\infty F_s[f(x)]F_s[g(x)]ds = \int_0^\infty f(x)g(x)dx$$

$$(iv)\ \int_0^\infty [F_s[f(x)]]^2\, ds = \int_0^\infty [f(x)]^2\, dx$$

21. Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ using transforms techniques.

**Solution:**

W.k.t $F_c\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}$

and $F_c\left[e^{-bx}\right] = \sqrt{\frac{2}{\pi}}\frac{b}{s^2+b^2}$

Now,

$$\begin{aligned}
\int_0^\infty F_c[f(x)]F_c[g(x)]ds &= \int_0^\infty f(x)g(x)dx \\
\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}\sqrt{\frac{2}{\pi}}\frac{b}{s^2+b^2}ds &= \int_0^\infty e^{-ax}e^{-bx}dx \\
\frac{2ab}{\pi}\int_0^\infty \frac{1}{(s^2+a^2)(s^2+b^2)}ds &= \int_0^\infty e^{-(a+b)x}dx \\
\int_0^\infty \frac{1}{(s^2+a^2)(s^2+b^2)}ds &= \frac{\pi}{2ab}\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty \\
&= \frac{\pi}{-2ab(a+b)}[0-1] \\
&= \frac{\pi}{2ab(a+b)} \\
\therefore \int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} &= \frac{\pi}{2ab(a+b)}
\end{aligned}$$

22. Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$ using transforms techniques.

**Solution:**

W.k.t $F_c\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}$

Now, $$\int_0^\infty [F_c[f(x)]]^2\, ds = \int_0^\infty [f(x)]^2\, dx$$

$$\begin{aligned}
\int_0^\infty \left(\sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}\right)^2 ds &= \int_0^\infty \left(e^{-ax}\right)^2 dx \\
\frac{2a^2}{\pi}\int_0^\infty \frac{1}{(s^2+a^2)^2} ds &= \int_0^\infty e^{-2ax} dx \\
\int_0^\infty \frac{1}{(s^2+a^2)^2} ds &= \frac{\pi}{2a^2}\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty \\
&= \frac{\pi}{2a^2(-2a)}[0-1] \\
&= \frac{\pi}{4a^3} \\
\therefore \int_0^\infty \frac{dx}{(x^2+a^2)^2} &= \frac{\pi}{4a^3}
\end{aligned}$$

23. Evaluate $\int_0^\infty \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)}$ using transforms.

**Solution:**

W.k.t $F_s\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}$

and $F_s\left[e^{-bx}\right] = \sqrt{\frac{2}{\pi}}\frac{s}{s^2+b^2}$

Now,

$$\begin{aligned}
\int_0^\infty F_s[f(x)]F_s[g(x)]ds &= \int_0^\infty f(x)g(x)dx \\
\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}\sqrt{\frac{2}{\pi}}\frac{s}{s^2+b^2} ds &= \int_0^\infty e^{-ax}e^{-bx}dx \\
\frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)} ds &= \int_0^\infty e^{-(a+b)x}dx \\
\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)} ds &= \frac{\pi}{2}\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty \\
&= \frac{\pi}{-2(a+b)}[0-1] \\
&= \frac{\pi}{2(a+b)} \\
\therefore \int_0^\infty \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} &= \frac{\pi}{2(a+b)}
\end{aligned}$$

24. Evaluate $\int_0^\infty \frac{x^2 dx}{(x^2+a^2)^2}$ using Parseval's identity.

**Solution:**

W.k.t $F_s\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}$

Now,

$$\int_0^\infty \left[F_s[f(x)]\right]^2 ds = \int_0^\infty \left[f(x)\right]^2 dx$$

$$\int_0^\infty \left(\sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}\right)^2 ds = \int_0^\infty \left(e^{-ax}\right)^2 dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)^2}ds = \int_0^\infty e^{-2ax}dx$$

$$\int_0^\infty \frac{s^2}{(s^2+a^2)^2}ds = \frac{\pi}{2}\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty$$

$$= \frac{\pi}{2(-2a)}[0-1]$$

$$= \frac{\pi}{4a}$$

$$\therefore \int_0^\infty \frac{x^2dx}{(x^2+a^2)^2} = \frac{\pi}{4a}$$

25. Evaluate $\int_0^\infty \frac{dx}{(x^2+4)(x^2+25)}$ using transforms techniques.

26. Evaluate $\int_0^\infty \frac{dx}{(x^2+1)^2}$ using transforms Parseval's identity method.

27. Evaluate $\int_0^\infty \frac{\lambda^2 d\lambda}{(\lambda^2+1)(\lambda^2+4)}$ using Parseval's identity.

28. Evaluate $\int_0^\infty \frac{x^2dx}{(x^2+1)^2}$ using Parseval's identity.

## Part-A

1. State Fourier integral theorem.

   Solution:

   **Fourier integral theorem**

   If f(x) is a given function defined in $(-l, l)$ and satisfies the following conditions,

   - f(x) is well defined and single valued except at finite number of points in $(-l, l)$
   - f(x) is periodic in $(-l, l)$
   - f(x) and $f'(x)$ are piecewise continuous in $(-l, l)$
   - $\int\limits_{-\infty}^{\infty} |f(x)|dx$ converges.

   then $f(x) = \frac{1}{\pi} \int\limits_{0}^{\infty} \int\limits_{-\infty}^{\infty} f(t) \cos \lambda(t-x) dt d\lambda$

2. State and prove Change of scale property for Fourier Transform.

   Statement:

   If $F[f(x)] = F(s)$ then $F[f(ax)] = \frac{1}{a} F\left(\frac{s}{a}\right), a > 0$

   Proof :

   The Fourier transform of $f(x)$ is

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(ax) e^{isx} dx$$

   Put $ax = t \Rightarrow adx = dt$ If $x = -\infty \Rightarrow t = -\infty$

   $dx = \frac{dt}{a}$ If $x = \infty \Rightarrow t = \infty$

$$\therefore F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(t) e^{is\left(\frac{t}{a}\right)} \frac{dt}{a}$$

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(t) e^{i\left(\frac{s}{a}\right)t} dt$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right)$$

3. State and prove the Shifting Property.

   If $F[f(x)] = F(s)$ then $F[f(x-a)] = e^{ias} F(s)$.

   Also $F[e^{iax} f(x)] = F(s+a)$ and $F[e^{-iax} f(x)] = F(s-a)$

   Proof :

The Fourier transform of $f(x)$ is

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{isx} dx$$

Put $x - a = t$ If $x = -\infty \Rightarrow t = -\infty$

$\Rightarrow dx = dt$ If $x = \infty \Rightarrow t = \infty$

$$\therefore F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{is(t+a)} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} e^{ias} dt$$

$$= e^{ias} F(s)$$

$$\text{Now } F[e^{iax} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s+a)x} dx$$

$$= F(s+a)$$

$$\text{Now } F[e^{-iax} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iax} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s-a)x} dx$$

$$= F(s-a)$$



4. If $F[f(x)] = F(s)$ then $F[f(x) \cos ax]$= ?

Proof :

The Fourier transform of $f(x)$ is

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$F[f(x) \cos ax] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos ax e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left( \frac{e^{iax} + e^{-iax}}{2} \right) e^{isx} dx$$

$$= \frac{1}{2} \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{iax} e^{isx} dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iax} e^{isx} dx \right]$$

$$
\begin{aligned}
&= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(s+a)x}dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(s-a)x}dx\right] \\
&= \frac{1}{2}[F(s+a)+F(s-a)]
\end{aligned}
$$

5. Find the Fourier transform of $f(x) = \begin{cases} e^{ikx}, & a < x < b; \\ 0, & x \le a \text{ or } x > b \end{cases}$

   Solution:

   The Fourier transform of $f(x)$ is

$$
\begin{aligned}
F(s) &= F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{a}^{b} e^{ikx}e^{isx}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{a}^{b} e^{i(s+k)x}dx \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{i(s+k)x}}{i(s+k)}\right]_a^b \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{i(s+k)}\left[e^{i(s+k)b} - e^{i(s+k)a}\right]
\end{aligned}
$$

6. Find $F_c[xf(x)]$ and $F_s[xf(x)]$

   Solution:

$$
\begin{aligned}
\text{W.K.T} \qquad F_c[f(x)] &= \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx ds \\
\frac{d}{ds}F_c[f(x)] &= \sqrt{\frac{2}{\pi}}\frac{d}{ds}\int_0^{\infty} f(x)\cos sx ds \\
&= \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)(-\sin sx)x ds \\
&= -F_s[xf(x)] \\
F_s[xf(x)] &= -\frac{d}{ds}F_c[f(x)]
\end{aligned}
$$

$$
\begin{aligned}
\text{W.K.T} \qquad F_s[f(x)] &= \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx ds \\
\frac{d}{ds}F_s[f(x)] &= \sqrt{\frac{2}{\pi}}\frac{d}{ds}\int_0^{\infty} f(x)\sin sx ds \\
&= \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx x ds \\
&= F_c[xf(x)]
\end{aligned}
$$

$$F_c[xf(x)] = \frac{d}{ds}F_s[f(x)]$$

7. Find the function $f(x)$ whose sine transform is $\dfrac{e^{-as}}{s}$

Solution:

Given: $F_s(s) = \dfrac{e^{-as}}{s}$

The Inverse Fourier Sine transform is

$$\begin{aligned}
f(x) &= \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin sx ds \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-as}}{s}\sin sx ds
\end{aligned}$$

Differentiating w.r.t 'x'

$$\begin{aligned}
\frac{d[f(x)]}{dx} &= \sqrt{\frac{2}{\pi}}\frac{d}{dx}\left[\int_0^\infty \frac{e^{-as}}{s}\sin sx ds\right] \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-as}}{s}\frac{\partial}{\partial x}(\sin sx)ds \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-as}}{s}\cos sx \cdot sds \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-as}\cos sx ds \\
\frac{d[f(x)]}{dx} &= \sqrt{\frac{2}{\pi}}\frac{a}{a^2+x^2} \\
f(x) &= \sqrt{\frac{2}{\pi}}\int \frac{a}{a^2+x^2}dx \\
&= a\sqrt{\frac{2}{\pi}}\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) \\
&= \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{a}\right)
\end{aligned}$$

8. Find the Fourier Sine transform of $\dfrac{1}{x}$

Solution:

The Fourier Sine transform is

$$\begin{aligned}
F_s[f(x)] &= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx ds \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x}\sin sx ds \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin sx}{x}ds \\
&= \sqrt{\frac{2}{\pi}}\frac{\pi}{2}
\end{aligned}$$

$$= \sqrt{\frac{\pi}{2}}$$

9. State the convolution theorem and Parseval's identity for Fourier Transfrom.

   Solution:

   Convolution Theorem:

   $$\text{If } F[f(x)] = F(s) \text{ and } F[g(x)] = G(s) \text{ then } F[f(x) * g(x)] = F(s) \cdot G(s)$$

   Parseval's identity:

   $$\text{If } F[f(x)] = F(s) \text{ then } \int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

## Other important questions:

1. Verify the convolution theorem under Fourier transform from $f(x) = g(x) = e^{-x^2}$.

   Solution:

   The Convolution of $f(x)$ and $g(x)$ is

   $$F[f(x) * g(x)] = F(s)G(s)..........(1)$$

   where $F[f(x)] = F(s)$ and $F[g(x)] = G(s)$

   To find $F(s)G(s)$

   $$\text{Given } f(x) = e^{-x^2} \text{ and } g(x) = e^{-x^2}$$

   W.K.T, $$F[e^{-a^2x^2}] = \frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}}$$

   Put $a = 1$

   $$F[e^{-x^2}] = \frac{1}{\sqrt{2}} e^{-\frac{s^2}{4}} = F(s)$$

   Similarly $$F[e^{-x^2}] = \frac{1}{\sqrt{2}} e^{-\frac{s^2}{4}} = G(s)$$

   Now $$F(s)G(s) = \frac{1}{\sqrt{2}} e^{-\frac{s^2}{4}} \frac{1}{\sqrt{2}} e^{-\frac{s^2}{4}}$$

   $$F(s)G(s) = \frac{1}{2} e^{-\frac{s^2}{2}}........(2)$$

   To find $F[f(x) * g(x)]$

$$\begin{aligned}
f(x) * g(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)g(x-t)dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} e^{-(x-t)^2} dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(t^2+(x-t)^2)} dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(t^2+x^2-2xt+t^2)} dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(2t^2-2xt+x^2)} dt
\end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(2t^2-2xt+x^2)} e^{-\frac{x^2}{2}} e^{\frac{x^2}{2}} dt$$

$$= \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(2t^2-2xt+x^2-\frac{x^2}{2})} dt$$

$$= \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2\left(t^2-2\frac{x}{2}t+\frac{x^2}{2}-\frac{x^2}{4}\right)} dt$$

$$= \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2\left(t^2-2\frac{x}{2}t+\frac{x^2}{4}\right)} dt$$

$$= \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2\left(t-\frac{x}{2}\right)^2} dt$$

Put $u = \sqrt{2}\left(t - \frac{x}{2}\right)$ $\Rightarrow du = \sqrt{2}dt$

if $t = -\infty$ $\Rightarrow u = -\infty$

if $t = \infty$ $\Rightarrow u = \infty$

$$f(x) * g(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} \frac{du}{\sqrt{2}}$$

$$= \frac{e^{-\frac{x^2}{2}}}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du$$

$$= \frac{e^{-\frac{x^2}{2}}}{2\sqrt{\pi}} \sqrt{\pi}$$

$$= \frac{e^{-\frac{x^2}{2}}}{2}$$

$$\text{Now } F[f(x) * g(x)] = F\left[\frac{e^{-\frac{x^2}{2}}}{2}\right]$$

$$= \frac{1}{2} F\left[e^{-\frac{x^2}{2}}\right]$$

$$= \frac{1}{2} e^{-\frac{s^2}{2}} \ldots\ldots\ldots.(3)$$

From (2) and (3),

$$F[f(x) * g(x)] = F(s)G(s)$$

Hence Convolution theorem is verified.

Note:

The Convolution of two functions $f(x)$ and $g(x)$ is

$$f(x) * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)g(x-t)dt$$

2. State and prove the Convolution and Parseval's identity Theorem.

   Convolution Theorem:

   If $F[f(x)] = F(s)$ and $F[g(x)] = G(s)$ then $F[f(x) * g(x)] = F(s) \cdot G(s)$

Proof:

$$\begin{aligned}
F[f(x)*g(x)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)*g(x)e^{isx}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)g(x-t)dt\right]e^{isx}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x-t)e^{isx}dx\right]f(t)dt \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x-t)]f(t)dt \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]e^{ist}f(t)dt \\
&= F[f(x)]F[g(x)] \\
&= F(s)G(s)
\end{aligned}$$

Parseval's identity:

If $F[f(x)] = F(s)$ then $\int_{-\infty}^{\infty} |F(s)|^2ds = \int_{-\infty}^{\infty} |f(x)|^2dx$

Proof:

By Convolution Theorem

$$\begin{aligned}
F[f(x)*g(x)] &= F(s)G(s) \\
f(x)*g(x) &= F^{-1}[F(s)G(s)] \\
\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)g(x-t)dt &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)G(s)e^{-isx}ds \\
\text{Put } x = 0 & \\
\int_{-\infty}^{\infty} f(t)g(-t)dt &= \int_{-\infty}^{\infty} F(s)G(s)ds \\
\text{Put } g(-t) = \overline{f(t)} \text{ and } G(s) = \overline{F(s)} & \\
\int_{-\infty}^{\infty} f(t)\overline{f(t)}dt &= \int_{-\infty}^{\infty} F(s)\overline{F(s)}ds \\
\int_{-\infty}^{\infty} |f(t)|^2dt &= \int_{-\infty}^{\infty} |F(s)|^2ds
\end{aligned}$$

$$\Rightarrow \int_{-\infty}^{\infty} |F(s)|^2ds = \int_{-\infty}^{\infty} |f(x)|^2dx, \qquad \text{Put } t = x$$

3. Find the Fourier integral representation of $f(x)$ defined as $f(x) = \begin{cases} 0, & \text{for } x < 0 \\ \frac{1}{2} & \text{for } x = 0 \\ e^{-x} & \text{for } x > 0 \end{cases}$

Verify the representation directly at the point $x = 0$.

Solution:

The Fourier transform is

$$\begin{aligned} F(s) &= F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x} (\cos sx + i \sin sx) dx \\ &= \frac{1}{\sqrt{2\pi}} \left[ \int_{0}^{\infty} e^{-x} \cos sx dx + i \int_{0}^{\infty} e^{-x} \sin sx dx \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[ \frac{1}{s^2+1} + i \frac{s}{s^2+1} \right] \\ &= \frac{1}{\sqrt{2\pi}} \frac{1+is}{s^2+1} \end{aligned}$$

The Inverse Fourier transform is

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1+is}{s^2+1} (\cos sx - i \sin sx) ds \\ &= \frac{2}{2\pi} \int_{0}^{\infty} \frac{\cos sx + s \sin sx}{s^2+1} ds \\ &= \frac{1}{\pi} \int_{0}^{\infty} \frac{\cos sx + s \sin sx}{s^2+1} ds \end{aligned}$$

Verification:

Put $x = 0$,

$$\begin{aligned} f(0) &= \frac{1}{\pi} \int_{0}^{\infty} \frac{1}{s^2+1} ds \\ &= \frac{1}{\pi} [\tan^{-1} x]_0^{\infty} \\ &= \frac{1}{\pi} \frac{\pi}{2} \\ f(0) &= \frac{1}{2} \end{aligned}$$